Functional Analysis 13 November 2012

1.1 Let $M$ be a closed linear subspace of $H$. Let $u \in H$. Then $u = P_m f$ if characterized by $u \in M$ and $(f-u, v) = 0$ for all $v \in M$ and $P_m f$ is linear.

$M$ convex, closed

Let $u \in M$ and $(f-u, v-u) \leq 0$ for all $v \in M$

Proof: $M$ convex and closed

$\exists! u = P_m f \text{ s.t. } \forall v \in M (f-u, v-u) \leq 0$

$t v \in M \text{ for } (f-u, tv-u) \leq 0$

$0 \leq t(f-u, v) \leq (f-u, u) \Rightarrow t(f-u, v) \leq (f-u, u) \forall t \in \mathbb{R}$

Let $v = 2u \in M$ then $(f-u, v) = 0$
Conversly \( u \in M \) st. \( \forall v \in M \quad (f-u,v)=0 \)
\[ v \rightarrow v-u \]
\[ \Rightarrow (f-u,v-u)=0 \leq 0 \quad \forall v \in M \]

So this is a projection.
\[ \forall v \in M \]
\[ (f_1-P_Mf_1,v)=0 \]
\[ (f_2-P_Mf_2,v)=0 \]
So \( \forall v \in M \)
\[ (f_1+f_2-P_M(f_1+f_2),v) \]
So \( P_M(f_1+f_2)=P_Mf_1+P_Mf_2 \)

**Riesz Representation Theorem**

H is a Hilbert space
\[ \forall \varphi \in \mathcal{H} \quad \exists ! \; u \in \mathcal{H} \quad \forall f \in \mathcal{H} \quad \langle \varphi, f \rangle = (f,u) \]

That is, the evaluation of \( \varphi \) at \( f \) is the inner product of \( f \) with a (fixed) vector.

**Proof**
\[ M = \{ x \in \mathcal{H} : \varphi(x) = 0 \} \]

\( M \) is a closed linear subspace of \( \mathcal{H} \)
if \( M = \mathcal{H} \) the result is trivial, \( \varphi \equiv 0 \), take \( u = 0 \)
Suppose then \( M \neq \mathcal{H} \)

Claim \( \exists g \in \mathcal{H}, \| g \|=1 \quad \forall v \in M \quad (g,v)=0 \)

\( \exists g_0 \in \mathcal{H} \setminus M \)

Let \( g_0 = P_M g_0 \) / Let \( g = \frac{g_0-g_1}{\|g_0-g_1\|} \)
\[ \|g_0-g_1\| > 0 \quad \text{since} \quad g_0 \notin M, \; g_1 \in M. \]
\[ (g, v) = \frac{1}{\|q_0 - q_1\|} (q_0 - q_1, v) = \frac{1}{\|q_0 - q_1\|} (g_0 - P_M g_0, v) \]
\[ = 0 \quad \forall v \in M \text{ by corollary } q_1.1 \]

Take \( u \equiv g \)

\[ \langle f, g \rangle = (f, v) \quad \forall f \in H \]

Given \( f \in H \Rightarrow \exists u_1 \in M \ u_2 \in M \text{ s.t. } f = u_1 + u_2 \)

Pick \( \lambda, f \in H \text{ s.t. } v = f - \lambda g \in M \)

\[ \phi(v) = \phi(f) - \lambda \phi(g) = 0 \]

\[ u_2 = \frac{\phi(f)}{\phi(g)} g \]

Let us find \( \lambda \)

\[ (g, v) = (g, f) - \lambda \]
\[ \| \quad 0 \quad \text{ thus} \]
\[ \text{ since } v \in M \quad (g, f) = \lambda \]

\[ (v, g) = (f, g) - \lambda \]
\[ \| \quad 0 \quad \text{ thus} \]
\[ \lambda = \frac{\phi(f)}{\phi(g)} \]

\[ \phi(v) = \phi(f) - \lambda \phi(g) = 0 \]

\[ f = v + \lambda g \]

\[ \phi(f) = \phi(v) + \lambda \phi(g) = \lambda \phi(g) = \phi(f) \phi(g) = (f, \frac{\phi(g) g}{v}) \]
**Banach Fixed Point Theorem**

\((E, d)\) is a complete metric space

\(T : E \to E\) which is a contraction, i.e. \(\exists K < 1 \text{ s.t. } \forall x, y \in E \quad d(Tx, Ty) \leq K d(x, y)\)

\(\exists ! x \in E \text{ s.t. } Tx = x\)

**Proof**

Pick \(x_0 \in E\) and let \(x_n = T^{n-1} x_0\)

Claim that \((x_n)\) is Cauchy \((n < m)\)

\[d(x_n, x_m) = \sum_{j=m}^{n} d(T^{j-1} x_0, T^{j-1} x_0) \leq \sum_{j=m}^{n} K^{j-1} d(Tx_0, x_0)\]

\[\leq K^{m-1} \sum_{j=0}^{\infty} K^j d(x_0, x_0)\]

So \(d(x_n, x_m) \to 0\) as \(n, m \to \infty\)

Thus \(d(x_n, x_m) \to 0\) as \(n, m \to \infty\)

Therefore by completeness

\(x_n \to x\)

But \(T\) is (Lipschitz) continuous, so \(Tx_n \to Tx\)

\[T x_n = T^{n+1} x_0 = x_{n+1} \to x = Tx\]

\(\Rightarrow x_1 = x_2\)
New notation:

\[ a : H^1 \rightarrow \mathbb{R} \text{ is bilinear.} \]

i) \( a \) is continuous if \( \exists C > 0 \forall u,v \in H \quad a(u,v) \leq C \| u \| \| v \| \)

ii) \( a \) is coercive if \( \exists \alpha > 0 \forall u \in H \quad a(u,u) \geq \alpha \| u \|^2 \)

Typical Example

\( (a_{ij}(x))_{1 \leq i,j \leq n} \) matrix satisfying \( \sum_{i,j=1}^{n} a_{ij}(x) \overline{x}_i x_j \geq \delta \| x \|^2 \)

\( \forall x \in \mathbb{R}^n, b \in \mathbb{R}^n \) (that is, \( \alpha = \text{uniformly elliptic} \)

\( H = W^{1,2}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \mid u, \partial u \in L^2(\Omega) \right\} \)

\( \| u \|_{H^1(\Omega)} = \left( \| u \|^2_{L^2(\Omega)} + \| \partial u \|^2_{L^2(\Omega)} \right)^{1/2} \)

\( (u,v) = \int_{\Omega} u v \, dx + \sum_{i=1}^{n} \int_{\Omega} \partial x_i u \partial x_i v \, dx \)

This space is also complete, thus a Hilbert space.

\[ a(u,v) = \int_{\Omega} u v \, dx + \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \partial x_i u(x) \partial x_j v(x) \, dx \]

\( a \) is continuous in \( W^{1,2}(\Omega) \) and is coercive.

\( u \) is used to solve the PDE

\[ \sum_{i,j=1}^{n} \partial x_i \left( a_{ij}(x) u x_j \right) = f \]
Theorem (Stampacchia)

Let \( a(u, v) \) be continuous & coercive bilinear form.

Let \( \mathcal{H} \) be nonempty, convex, and closed.

Then \( \forall \psi \in \mathcal{H}^* \exists! u \in \mathcal{H} s.t. \forall v \in \mathcal{H}
\[
a(u, v-u) \geq \langle \psi, v-u \rangle
\]

If \( a \) is symmetric, then \( u \) is characterized by

\( u \in \mathcal{H} \)
\[
\frac{1}{2} a(u, u) - \langle \psi, u \rangle = \min_{v \in \mathcal{H}} \left\{ \frac{1}{2} a(v, v) - \langle \psi, v \rangle \right\}
\]

Proof

* By Riesz representation theorem \( \exists! f \in \mathcal{H}^* s.t. \langle \psi, v \rangle = \langle f, v \rangle \forall v \in \mathcal{H}^* \)

* Fix \( u \in \mathcal{H} \), the map \( v \mapsto a(u, v) \)

is linear and continuous on \( \mathcal{H} \).

i.e., it is in \( \mathcal{H}^* \) so by RRT \( \exists! A u \in \mathcal{H} s.t. \langle Au, v \rangle = a(u, v) \)

Genuinely \( u \mapsto Au \) is linear because \( f \), \( u_1 \), \( u_2 \)
\[
v \mapsto a(u_1 + t u_2, v) = (Au_1, v) + t (Au_2, v) \]

\( u \mapsto a(u_1 + u_2, v) = (Au_1 + Au_2, v) \) but also equals

\( u \mapsto A(u_1 + u_2, v) \) says the representing vector is unique so \( Au_1 + Au_2 = A(u_1 + u_2) \)

\[
|a(u, v)| \leq \|Au\| \|v\|
\]

\[
\|Au\| = \sup_{\|v\|=1} |(Au, v)| \leq C \|u\| \|v\|
\]

\[
|a(u, u)| = |(Au, u)| \geq \|u\|^2 \forall u \in \mathcal{H}
\]
\[ a(u, v-u) = (Au, v-u) \]

To find \( u \in K \) s.t. \( \forall v \in K \)

\[ (Au, v-u) \geq (f, v-u) \]

find \( u \) by projection + fixed point theorem.