WEAK* Topology of $E^*$

For $x \in E$, define $\varphi_x : E^* \rightarrow \mathbb{R}$, $\varphi_x(f) = \langle f, x \rangle$

For $x \in E$, $\varphi_x$ is a linear map for each $x \in E$.

Apply the construction: $X \mapsto E^*$

$I = E$

$Y_i = \mathbb{R}$

$\{\varphi_j \}_{j \in I}$

An open set in $E^*$ in the weak* topology is of the form $\varphi_x^{-1}(\text{open})$

This is the weak* topology of $E^*$, denoted $\tau(E^*, E)$, the coarsest topology such that every $\varphi_x$ is continuous.

Goal: Bourbaki- Alaoglu: the unit ball in $E^*$, $B_{E^*} = \{f \in E^* : \|f\| \leq 1\}$ is compact in the weak* topology.

Application of this theorem, e.g. to solve an equation, construct a sequence of approximate solutions $\{u_n\}$, the solutions are bounded (in some norm), they since they are in a compact set, we can extract a convergent subsequence in some norm, $\{u_{n_k}\} \rightarrow u$, which is then shown to be a solution to the equation.
Proposition: \( E^* \) with the topology \( \sigma(E^*, E) \) is a Hausdorff space.

Proof: \( f_1, f_2 \in E^* \), \( f_1, f_2 \) want \( f_1 \in O_1 \) open, \( f_2 \in O_2 \) open, \( O_1 \cap O_2 \neq \emptyset \)

\( f_1 + f_2 \Rightarrow \exists x \in E \) \( \langle f_1, x \rangle \neq \langle f_2, x \rangle \). Wlog \( \langle f_1, x \rangle < \langle f_2, x \rangle \)

pick \( \alpha \) in between \( \langle f_1, x \rangle < \alpha < \langle f_2, x \rangle \)

\[ O_1 = \{ f \in E^* : \langle f, x \rangle < \alpha \} = \mathcal{O}^{-1}(\alpha, \infty) \]

\[ O_2 = \{ f \in E^* : \langle f, x \rangle > \alpha \} = \mathcal{O}^{-1}(\alpha, \infty) \]

N.B. Unlike the weak \( \sigma(E, E^*) \) topology, we don't need Hahn-Banach for separation, but only \( f_1 + f_2 \)

\( f_n \) converges to \( f \) in the weak* topology is written \( f_n \overset{w^*}{\to} f \)

Neighborhoods in the weak* topology

\( f_0 \in E^* \)

\( \{ x_1, \ldots, x_k \} \subseteq E \), \( \varepsilon > 0 \)

\[ V(x_1, \ldots, x_n; \varepsilon) = \{ f \in E^* : \| f(x_i) - f_0(x_i) \| < \varepsilon, \ i = 1, \ldots, n \} \]

is a basis of the weak* topology

- \( f_n \overset{w^*}{\to} f \ \text{in} \ \sigma(E^*, E) \iff \forall x \in E \ \langle f_n, x \rangle \to \langle f, x \rangle \)
- \( f_n \overset{w^*}{\to} f \text{, i.e. } \| f_n - f \|_{w^*} \to 0 \Rightarrow f_n \overset{w^*}{\to} f \)
- \( f_n \overset{w^*}{\to} f \Rightarrow \exists M \forall n \| f_n \| \leq M \); \( \| f \| \leq \liminf \| f_n \| \)
Bourbaki - Alaoglu

\[ \overline{B}_{E^*} = \{ f \in E^* : \|f\|_* \leq 1 \} \] is compact in \( \sigma(E^*, E) \)

\( \text{n.b. } E \text{ is an m.v.s. not neces. Banach} \)

**Proof**

\[ Y = \mathcal{F}L^E = \{ \text{all maps from } E \text{ to } \mathbb{R} \} \]

\( \omega \in Y \quad \phi : E \to \mathbb{R}, \quad \omega = (\omega_x)_{x \in E} \)

Give \( Y \) the product topology from the usual topology on \( \mathbb{R} \)

\[ E^* \subseteq Y \]

define \( \Phi : E^* \to Y \) \( \Phi(f) = (\langle f, x \rangle)_{x \in E} \) i.e. \( \Phi : E^* \to Y \)

\( E^* \) has the \( \sigma(E^*, E) \) topology

\( Y \) has the product topology

\( \Phi : E^* \to Y \) is continuous, since if \( f_n \to f \) \( \iff \forall x \in E \langle f_n, x \rangle \to \langle f, x \rangle \)

\( \iff \Phi(f_n) \to \Phi(f) \)

Now \( \Phi(E^*) \subset Y \) so take \( \Phi^{-1} : \Phi(E^*) \to E^* \)

\( \Phi^{-1} \) is 1-to-1

\( \Phi^{-1} \) continuous with topology of \( Y \) to weak* topology of \( E^* \)

Given \( x \in E \) the map \( \omega \mapsto \langle \Phi^{-1}(\omega), x \rangle \quad \omega \in \Phi(E^*) \)

\[ \overline{\Phi(E^*)} \overset{\Phi^{-1}}{\to} E^* \]

1. We have that \( \Phi^{-1} \) is continuous \( \iff \forall x \in E \Phi^{-1} \) continuous
\[ \langle \phi^*(w), x \rangle = \langle f, x \rangle = \omega_x \]
\[ \omega = \phi(f) \]

\[ \overline{\phi(E^*)} \leftrightarrow E^* \quad \text{so} \quad \overline{\phi(E^*)} \text{ is homeomorphic to } E^* \]

To show that \( \overline{\phi(E^*)} \) is compact in \( \mathcal{C}(E^*, E) \)

\[ K = \phi(\overline{\phi(E^*)}) \text{ is compact in } \phi(E^*) \subset Y \]

So \( K := \overline{\phi(E^*)} = \{ \omega \in Y : |\omega_x| \leq \|x\|, \omega_x = \omega_x^1 + \omega_x^2, \forall x \in E \forall \alpha \in \mathbb{R} \} \)

\[ \phi(f) = (\langle f, \lambda \rangle)_{\lambda \in E} \]

\[ \omega = (\omega_x)_{x \in E} \]

\[ \omega_x \in \mathbb{R} \]

\[ K_1 = \{ \omega \in Y : |\omega_x| \leq \|x\| \} \]

\[ K_2 = \{ \omega \in Y : \omega \text{ is linear} \} \]

Claim \( K_1 \) is compact and \( K_2 \) is closed, and \( K = K_1 \cap K_2 \)

so by Hausdorff compactness,

\[ K_1 = \prod_{x \in E} \left[ -\|x\|, \|x\| \right] \]

Tychonoff's theorem

Product of compact sets is compact

and

Extension of choice
\[ A_{xy} = \{ \omega \in Y : \omega_{xy} - \omega_x - \omega_y = 0 \} \]

\[ B_{\lambda, x} = \{ \omega : \omega_{\lambda x} - \lambda \omega_x = 0 \} \]

But,

\[ K_\lambda = \bigcap_{x,y, \lambda} A_{xy} \cap B_{\lambda, x} \text{ is closed.} \]

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**Example 3.9**

Let \( E \) be a normed space and \( M \) a subspace of \( E \), \( f_0 \in E^* \).

Consider \( M^\perp = \{ f \in E^* : \langle f, x \rangle = 0 \text{ for all } x \in M \} \).

\( M^\perp \) is a linear subspace of \( E^* \).

**Proof**

Existence \( \exists g_0 \in M^\perp \) s.t.

\[ \inf_{g \in M^\perp} \| f_0 - g \|_* = \| f_0 - g_0 \|_* \]

\[ \text{dist}(f_0, M^\perp) \]

**Proof**

\( M^\perp \) is closed in weak* topology and \( f_n \to f, f_n \in M^\perp \).

\[ \Rightarrow \langle f_n, x \rangle \to \langle f, x \rangle \quad \forall x \]

\[ \Rightarrow \forall x \in M \langle f_n, x \rangle = 0 \to \langle f, x \rangle \]

So by def. of infimum

\[ \inf_{g \in M^\perp} \| f_0 - g \|_* = \lim_{n \to \infty} \| f_0 - g_n \|_* \]

for some sequence \( (g_n) \in M^\perp \)
\[ \|g_n\|_* = \|f_0 - g_n\|_* + \|f_0\|_* \leq M + \text{some number} \quad \forall n \]

\[ g_n \in B_{\mathcal{E}}^+(M) \text{ is compact } \exists n_{k} \]

\[ \exists \quad g_{n_k} \to g \in M^+ \]

so \( \inf_{g \in M^+} \|f_0 - g\|_* \leq \|f_0 - g\|_* \) since \( g \in M^+ \)

\[ <f_0, g_{n_k}> \to <f_0, g, x> \]

\[ \|x\| \leq \|f_0 - g_{n_k}\|_* \|x\| \]

if \( \|x\| \leq 1 \) \Rightarrow \|x\| \leq \|f_0 - g_{n_k}\|_* \]

\[ |<f_0, g, x>| \leq \lim_{k \to \infty} \|f_0 - g_{n_k}\|_* \]

i.e. \( \|f_0 - g\|_* \leq \lim_{k \to \infty} \|f_0 - g_{n_k}\|_* = \inf_{g \in M^+} \|f_0 - g\|_* \)

So take \( g_0 = g \). \quad \square
Ex. 3.5:

E Banach

K ⊆ E is compact in strong topology (E, ‖ ‖)

{xn} ⊆ K s.t. xn → x in C(E, E*)

⇒ xn → x strongly in (E, ‖ ‖).

Proof

Suppose for contradiction xn → x. Then

∃ ε > 0, ∃ (nk) s.t. ‖xn_k - x‖ > ε.

Since K is compact in X, Xn_k → z strongly in Z.e.K

⇒ Xn_k → z weakly.

⇒ ‖Z - x‖ > ε by contradictory assumption.

But

Xn_k → Z

⇒ x

⟨f, xn_k⟩ → ⟨f, z⟩

⇒ ⟨f, x⟩

for every f ⇒ ⟨f, z - x⟩ = 0

⇒ Z - x = 0

or more directly - in Hausdorff topology, cannot have two limits for one sequence.