Weak Topologies

Let \( X \) be a space and \( \{ Y_i \} \) be topological spaces.
Let \( \varphi_i : X \to Y_i \).

We want to construct a topology on \( X \) such that all the \( \varphi_i \) are continuous.
That is, if \( W_i \subseteq Y_i \) is open, \( \varphi^{-1}_i(W_i) \) must be open in \( X \).

The topology on \( X \) is the weak topology if it is the "smallest" such on \( X \).

Let \( X \) and \( \{ U_A \}_{A \in \Lambda} \) be subsets of \( X \).

Find the smallest topology on \( X \) such that it contains \( \{ U_A \}_{A \in \Lambda} \).

1st Consider \( \mathcal{F} \subseteq \Lambda \) finite, and let \( \mathcal{F} = \{ \bigcap_{A \in \Lambda} U_A \} \).

2nd step Consider \( \mathcal{F} = \{ \bigcup_{A \in \Lambda} U_A \} \).

Let \( A \subseteq \Lambda \) be an arbitrary subset of \( \Lambda \).

Let \( U \subseteq \bigcup_{A \in \Lambda} U_A \).

i.e. \( U = \bigcap_{A \in \Lambda} U_A \).

Theorem: The finite intersection of members in \( \mathcal{F} \) belongs to \( \mathcal{F} \).

So \( \mathcal{F} \) is the smallest topology containing \( \{ U_A \}_{A \in \Lambda} \).
$X$

$$U_i \rightarrow \{ \varphi_i^{-1}(w_i) \} \quad \text{for } i \in I, \quad w_i \in Y_i \quad \text{open}$$

$T$ is the topology generated as above by the family $\{ \varphi_i^{-1}(w_i) \}$.

If $x \in X$, $w_i$ is open, $\varphi_i(x) \in W_i$, i.e. $x \in \varphi_i^{-1}(w_i)$

Then
$$\bigcap_{i \in F} \varphi_i^{-1}(w_i)$$

is a basis for the neighborhoods of $x$ for finite $F$.

**Proposition 2.1**

$x_n \rightarrow x$ in the topology $T$ iff $\forall i \in I, \varphi_i(x_n) \rightarrow \varphi_i(x)$

i.e. the topology is the pointwise topology of $x_n - x_n \rightarrow x$ if pointwise.$\varphi_i$.

**Proof**

$\varphi_i$ are continuous functions in the topology $T$.

$x_n \rightarrow x$ in $T$ then $\varphi_i(x_n) \rightarrow \varphi_i(x)$ by definition.

Fix $x$, let $U$ be a neighborhood of $x$.

$\Rightarrow \exists \ J \subset I$ finite and open set $V_i \subset Y_i$ s.t.

$x \in \bigcap_{i \in J} \varphi_i^{-1}(V_i) = U \quad \varphi_i(x) \in V_i$

Fix $i \in J$. Define $\varphi_i(x) \rightarrow \varphi_i(x) \quad \exists N_i \ni \forall n > N_i, \varphi_i(x_n) \in V_i \rightarrow x_n \in \varphi_i^{-1}(V_i) \quad \forall n \geq N_i$

Let $N = \max_{i \in J} N_i \Rightarrow \forall n > N$, $x_n \in \bigcap_{i \in J} \varphi_i^{-1}(V_i)$.
Thus \( X_n \in U \Rightarrow X_n \to x \text{ in } T \)

**Prop**

\( Z \) is a topological space \( \psi : Z \to X \) map

\( X \) has topology \( T \) via \( \{Y_i\} \)

Then \( \forall \psi \text{ is continuous } \Leftrightarrow \Phi^i_{\infty} \psi \psi_i \circ \psi \text{ is continuous for all } i. \)

**Proof**

\( \Rightarrow \) just because composition of continuous functions is continuous,

\( \Leftarrow \) if \( \psi_i^-(U) \) is open in \( X \), then \( \psi_i^-(U) \) is open in \( Z \)

\( 
\psi_i^-(U) \Rightarrow \bigcup_{\text{finite}} \Phi_i^-(U_i) \text{ open in } X_i 
\)

\( \psi^-(U) = \psi^- \left( \bigcup_{\text{finite}} \Phi_i^-(U_i) \right) = \bigcup_{\text{finite}} \psi_i^- \left( \Phi_i^-(U_i) \right) \)

\( \psi^-(U) \text{ open in } Z \Rightarrow \text{ continuous for } \)

Therefore \( \psi^-(U) \) is open. \( \Box \)

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**Constructing The Weak Topology Over E**

\( E \) a Banach space. Consider the linear functionals on \( E \to \mathbb{R} \)

Given \( f \in E^* \), define \( \Phi_f(x) = \langle f, x \rangle = f(x) \)

So we will perform the construction above with

\( X = E \)

\( Y_i = E \)

\( i \in E^* \)

\( y \in \mathbb{R} \)
DEF The weak topology of $E$, denoted $\sigma(E, E^*) = \tau$
with $I = E^*$, $\varphi_i = \varphi_f$.

By construction, we have $\sigma(E, E^*)$ is the smallest topology for which all continuous linear functionals on $E$ remain continuous. Notice $\sigma(E, E^*) \subset \text{Top}(E, I, \tau)$.

Proposition 4.1
$\sigma(E, E^*)$ is Hausdorff topology on $E$.

Proof
Given $\forall x_1, x_2 \in E \implies \exists U_1, U_2$ open in $\sigma(E, E^*)$,

$x_1 \in U_1$, $x_2 \in U_2$, $U_1 \cap U_2 = \emptyset$

$\{x_1\}, \{x_2\}$ are compact so by Hahn–Banach strict separation

$\exists f \in E^*$ and $\forall x \in E^*$ $f(x_1) < x < f(x_2)$

so the pre-images of $(-\infty, x)$, $(x, \infty)$ work as $U_1, U_2$

Proposition 4.2
Let $x_0 \in E$, $\epsilon > 0$, $f_1, \ldots, f_k \in E^*$, let

$V = V(f_1, \ldots, f_k, E) = \{x \in E \mid |\langle f_i, x - x_0 \rangle| < \epsilon \; i = 1, \ldots, k\}$

Then $V$ is a neighborhood of $x_0$ in $\sigma(E, E^*)$.

Therefore $\{V(f_i, \ldots, f_k, E)\}$ is a basis of the neighborhood of $x_0$.

Proof
$-\epsilon < \langle f_i, x - x_0 \rangle < \epsilon$

$\langle f_i, x \rangle - \epsilon < \langle f_i, x \rangle < \epsilon + \langle f_i, x_0 \rangle$

$a_i = \langle f_i, x_0 \rangle$

$V = \cap_{i=1}^k \varphi_i^{-1}(a_i - \epsilon, a_i + \epsilon)$
So $V$ is a finite intersection of open sets (preimage by a function of an interval) which all contain $x_0$.

Need to show $\{V\}$ is a basis.

Let $U$ be a neighborhood of $x_0$ in $\sigma(C,E,\mathbb{E}^*)$.

Now, $\left\{ \bigcap_{i \in F} f_i^{-1}(W_i) \right\}$, $W_i$ open in $\mathbb{E}$, is a basis of neighborhoods.

In particular, given $U$,

$v_{x_0} = x_0 \in \bigcap_{i \in F} f_i^{-1}(W_i) \subset U$

so $f_i(x_0) \in W_i \quad i \in F$

thus $B(f(x_0),\varepsilon) \subset W_i$ for some $\varepsilon > 0$ and uniform over $F$ since $F$ finite (take min).

$x_0 \in f_i^{-1}(a_i - \varepsilon, a_i + \varepsilon)$

so $x_0 \in \bigcap_{i \in F} f_i^{-1}(a_i - \varepsilon, a_i + \varepsilon) \subset \bigcap_{i \in F} f_i^{-1}(W_i) \subset U$

Def $x_n \rightharpoonup x$ weakly if $x_n \rightharpoonup x$ in the weak topology. Write $x_n \rightharpoonup x$.

Proposition

1) $x_n \rightharpoonup x \iff \forall f \in E^* \ f(x_n) \rightarrow f(x)$

2) $x_n \rightarrow x$ 'strongly' $\Rightarrow x_n \rightharpoonup x$

3) $x_n \rightarrow x \Rightarrow \|x_n\| \leq M \forall n \geq N$ and $\|x\| \leq \liminf \|x_n\|$

4) $x_n \rightarrow x$ and $f_n \rightarrow f$ in $E^*$ $\Rightarrow f(x_n) \rightarrow f(x)$
Proof

1) follows from prop p. 2.1.

3) Recall Corollary 2.4 consequence of Banach-Steinhaus

   If E is Banach, B subset st. \( \forall f \in E^* \)
   
   the set \( \{ \langle f, x \rangle : x \in B \} \) is bounded \( \Rightarrow \) B is bounded in E

   Take \( B = \{ x_n \} \)
   
   \( x_n \to x \) \( \langle f, x_n \rangle \to \langle f, x \rangle \) by (1) \( \forall f \in E^* \)

   \( \{ \langle f, x_n \rangle \} \) is bounded \( \Rightarrow \) \( \{ x_n \} \) is bounded

   \( \langle f, x \rangle = \lim \langle f, x_n \rangle \)

   \[ |\langle f, x_n \rangle| \leq \|f\| \|x_n\| \]

   \[ |\langle f, x_n \rangle| \leq \|f\| \limsup_{n \to \infty} \|x_n\| \]

   \( \|x\| = \sup_{\|f\| \leq 1} |\langle f, x \rangle| \)

   so

   \[ \sup_{\|f\| \leq 1} |\langle f, x_n \rangle| \leq \sup_{\|f\| \leq 1} \lim \|x_n\| = \liminf \|x_n\| \]

   (2) \[ |\langle f, x_n \rangle - \langle f, x \rangle| = |\langle f, x_n-x \rangle| \leq \|f\| \|x_n-x\| \to 0 \]

   (4) \[ |\langle f_n, x_n \rangle - \langle f, x \rangle| = |\langle f_n, x_n-x \rangle + \langle f_n, x-x \rangle - \langle f, x \rangle| \]

   \[ = |\langle f_n-f, x_n \rangle + \langle f, x_n \rangle - \langle f, x \rangle| \]

   \[ \leq |\langle f_n-f, x_n \rangle| + |\langle f, x_n \rangle - \langle f, x \rangle| \]

   \[ \to 0 \quad \text{since} \quad x_n \to x \]

   \[ \leq \|f_n-f\| \|x_n\| \to 0 \quad \text{since} \quad x_n \to x \]

   \[ 0 \cdot M = 0 \]
Proposition If $E$ has finite dimension

$$X_n \to x \iff X_n \to x \text{ strongly}$$

Better - in finite dimensions, the weak topology is the same as the strong norm topology.

We have

$$(E, \| \cdot \|) \subseteq \sigma(E, E^*) \text{ i.e. every open set in } \sigma \text{ is open in } E$$

need to show the other inclusion for finite dimensions.

G. Remark 4 p60 a discussion of weak to strong topologies

$$L^2[-\pi, \pi] \quad f_n(x) = e^{inx}$$

This sequence converges weakly to 0 i.e. $f_n \rightharpoonup 0$

Since

$$f : L^2 \to \mathbb{R} \text{ linear, continuous}$$

then

$$f(f) = \int_{-\pi}^{\pi} \overline{g(x)} f(x) \, dx$$

$$\Rightarrow \int_{-\pi}^{\pi} g(x) e^{inx} \, dx \to 0$$

for any $g(x) \in L^2[-\pi, \pi]$ via Riemann-Lebesgue

$$\int_a^b e^{inx} \, dx = \frac{e^{inx}}{in} \bigg|_a^b < \frac{b}{n} \to 0$$

so for any characteristic functions, thus for all simple functions, thus