Actuarial Present Value

State space \( S := \{0, 1, \ldots \} \) (countable, finite or not)

At time \( n \), the individual is in state \( n \).

When the individual is in state \( j \), receive payment \( P_j \) (think of this as a premium).

Also, another kind of payment \( B \) (benefit) \( B_j \) is paid whenever a transition from state \( i \) to \( j \) is made.

Start in state \( j \) at time \( n \). At time \( n+1 \) make payment of \( \$100 \) if still in state \( j \).

Take time discount rate \( i = 100\% \) / time.

So at time \( n+1 \), the present value is \( \frac{100}{1+i} \).

Define \( V := \frac{1}{1+i} \) the discount factor.

In general, the value of a payment \( x \) at time \( n+1 \) at time \( n \) is \( x \cdot V^n \) [APV]

In a Markov chain example:

Homogeneous \( Q := \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \) start in state 0, whenever

in state 1 pay \( P \). Calculate out for times \( n, n+1, n+2, n+3 \).

Discount factor is \( V := \frac{1}{1+i} \).
\[ \text{APV} \left( C^{(j)} \right) = \sum_{k=0}^{\infty} \left( k Q_n^{(j)} C^{(j)} v^k \right) \]

- \( C^{(j)} \): Future cash flows in state \( j \)
- \( k Q_n^{(j)} \): Probability of being in state \( j \) at time \( k \)

So for example above

\[ 0 + PV \left( \frac{1}{3} \right) + PV^2 \left( \frac{1}{3} + \frac{2}{3} \cdot \frac{3}{4} \right) + PV^3 \left( \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{3}{4} \right) + \ldots \]

- \( Q_n^{(i,j)} \): Probability of transitioning from state \( i \) to state \( j \)
- \( P_n^{(i, \text{stay})} \): Probability of staying in state \( i \)

E.g. payment \( p \) for as long as remains in state \( i \), then terminating

\[ \text{APV} = P \left[ 1 + PV \left( \frac{3}{4} \right) + PV^2 \left( \frac{3}{4} \right)^2 + \ldots + \left( PV \left( \frac{3}{4} \right) \right)^n + \ldots \right] = \frac{1}{1 - PV \left( \frac{3}{4} \right)} \]

Read Examples 5, 6
Example

Four states \{F, G, H, J\} \quad \pi = 0.9

\{0, 1, 2, 3\}

\[ S = 0 = F \]

This item \( x \) in state 0, get paid at the end of 3 years

if \( x \) is in state 0, amount \( \$500 \)

\[ Q = \begin{pmatrix}
0.2 & 0.8 \\
0.5 & 0.5 \\
0.75 & 0.25 \\
1 & 0
\end{pmatrix} \]

Timeline

\[ 0 \quad 1 \quad 2 \quad 3 \]

\[ 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \]

\[ 1 \rightarrow 1 \rightarrow -1 \rightarrow 3 \]

\[ 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \]

\[ 0 \rightarrow 1 \rightarrow 0 \rightarrow 0 \]

\[ 0 \rightarrow 0 \rightarrow 1 \rightarrow 0 \]

\[ 0 \rightarrow 1 \rightarrow 2 \rightarrow 0 \]