Probability
31 January 2011

Counting Process
Independent Increments
Stationary Increments

Jack from Mad River

disjoint intervals → independent distribution

intervals of equal length have equal distribution of value

\( \frac{N(t) - N(t_0)}{t - t_0} \sim \frac{N(s) - N(s_0)}{s - s_0} \) if \( t, t_0, s, s_0 \)

\( N(t) \) events in \( (0, t] \)

\( N(t) - N(s) \) number of events in \( (s, t] \)

with independent increments

\( P[N(s) = 5, \ N(t) - N(s) = 4] = P[N(s) = 5] P[N(t) - N(s) = 4] \)

Notice \( P[N(s) = 5, \ N(t) = 9] \neq P[N(s) = 5] P[N(t) = 9] \)

since intervals not disjoint

Poisson Distribution

Last time had a Poisson process

\( P[N(s) = n, \ N(t) = m] = P[X = n] \)

\( X \sim B(n, \lambda t) \) independent of \( \lambda \)

Example 3.11

Customers arrive according to Poisson Process at rate \( \lambda = 5/hr \)

Given 3 customers arrived in the first hour, find the probability

i) One customer arrived in the first 40 minutes
ii) At most 2 customers arrived in the first 40 minutes
iii) The third customer arrived in the last 20 minutes
iv) Exactly one customer arrived in the last 20 minutes

So i) \( N(t) \sim P(\lambda t) \) so \( N(1) = P(5) = e^{-5} \frac{5^5}{5!} \)

\( P[N(\frac{2}{3}) = 1 \mid N(1) = 3] = P[X = 1] \) where \( X \sim B(3, \frac{2}{3}) \)

\( - (\frac{3}{1})(\frac{2}{3})^1(1 - \frac{2}{3})^{-1} = a \) \( (\frac{1}{3})^2 = \frac{2}{9} \)
1) \( P[N(\frac{2}{3}) \leq x \mid N(1) = 3] \)

\[
\begin{align*}
P[N(\frac{2}{3}) = 3 \mid N(1) = 3] &= P[X = 3] \\
&= \left( \frac{2}{3} \right)^3 \left( 1 - \frac{2}{3} \right)^0 \\
&= \left( \frac{2}{3} \right)^3 = \frac{8}{27}
\end{align*}
\]

\( 8 + 19 = 27 \quad \checkmark \)

\[ P[N(t) = n_t \mid N(s) = n_s] \quad 0 < s < t \]

\[
= \frac{P[N(t) = n_t, N(s) = n_s]}{P[N(s) = n_s]} = \frac{P[N(t) - N(s) = n_t - n_s, N(s) = n_s]}{P[N(s) = n_s]}
\]

\( \text{by independent increments} \)

\[
= P[N(t) - N(s) = n_t - n_s] \cdot P[N(s) = n_s] \quad \text{by stationarity}
\]

\[
= \sim \mathcal{P} \left( \lambda (t-s) \right) = e^{-\lambda (t-s)} (\lambda (t-s))^{n_t-n_s} / (n_t-n_s)!
\]

The exponential distribution is uniquely memory-free among continuous random variables.

Exercise 4

Independent Bernoulli trials

\( \mathcal{P} \) - probability of success

\( p \), \( 1-p \) - failure

\( Y \) - number of trials until first failure

\[
P[Y = j] = p^{j-1} (1-p)
\]

You have observed that \( k \) first trials were all successes

\( Z \) - additional number of trials needed to get first failure

\[
P[Z = j \mid Y > n] = \frac{P[Y = n+j \mid Y > n]}{P[Y > n]} = \frac{P(Y = j)}{\sum_{n+j} P(Y = n+j)} = \frac{p^{n+j-1} (1-p)}{\sum_{n+j} p^{n+j} (1-p) \sum_{n+j}}
\]
\[ P_{\text{prob}} \]

\[ \sum_{k=1}^{\infty} p^{k-1} (1-p) = p^n \]

\[ \sum_{k=1}^{\infty} p^{k-1} (1-p) = \sum_{k=1}^{\infty} p^{k-1} - p^k = \]