Probability 26 January 2011

solutions for problems for graduate students to be returned.

Kevin is playing a game from 100 to 1000.

\[ X_1 \quad X_2 \]

distributed \( \text{Exp}(0.7) \) (time called away)

Tina will play after Kevin leaves until her mother calls \( n X_1 \) days.

distributed \( \text{Exp}(0.6) \)

when she plays, scores 10,000 points per hour

\[ Y_1 = \min(X_1, X_2) \quad Y_1 = X_1 \quad \text{or} \quad Y_1 = X_2 \]

\[ Y_2 = \max(X_1, X_2) \]

\[ E(Y_2 - Y_1 \mid Y_1 = X_1) \]

\[ E(Y_2 - Y_1 \mid Y_1 = X_2) \]

\[ 0 \quad \text{since no play to} \quad P(X_1 = X_2) = 0 \]

with to find \( E(Y_2 - Y_1 \mid Y_1 = X_1) P(Y_1 = X_1) + E(Y_2 - Y_1 \mid Y_1 = X_2) P(X_2 < X_1) \)

\[ P(X_2 < X_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2} \]

\[ E(Y_2 - Y_1 \mid Y_1 = X_2) \quad \text{expectation of } X_1 \quad \text{exp}(\lambda_2) \]

So expected playing time \[ \frac{\lambda_2}{\lambda_1 (\lambda_1 + \lambda_2)} \]

\[ \text{Page 59, \#40} \]
Poisson Process

1. Counting process $N(t)$

   has measured from an arbitrary time $0$
   counting appearances before on time $t$

Poisson Distribution

$t$ fixed, $N$ is variable

e.g. how many earthquakes per year,
   typos per page, etc.

$N \sim P(\lambda) =$

$P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}$ where $n \in \mathbb{N}$

$N(t)$ for counting process (a random variable evolving
   over time)

$N(t) =$ number of events occurring in $(0, t]$

$N(t) \geq 0$

$N(t) : \mathbb{R} \rightarrow \mathbb{Z}$, so has integer values

$N(t)$ is stationary

if $s < t$, then $N(t) - N(s)$ is number of events
   in interval $(s, t]$

Independent Increments

situation exists if when

$P(N(t) = n_1, N(t) - N(s) = n_2) = P[N(t) = n_1]P[N(t) - N(s) = n_2]$
STATIONARY INCREMENTS

\[0 \quad t_1 \quad t_2 \quad s \quad t \quad t_2 - t_1 = \varepsilon\]

Say STATIONARY INCREMENTS if
\[\mathbb{P}(N(\varepsilon) = \eta) = \mathbb{P}(N(t) - N(s) = \eta)\]

Def POISSON PROCESS

A counting process with independent & stationary increments, and
\[\mathbb{P}(N(t) = \eta) = e^{-\lambda t} \frac{(\lambda t)^\eta}{\eta!} \quad \eta = 0, 1, 2, \ldots\]

\[N(t) \sim \text{Pois}(\lambda t) \quad \lambda - \text{rate}\]

\(\lambda\) is constant (i.e. this is a homogeneous poisson process)

So
\[\mathbb{P}(N(t_2) - N(t_1) = \eta)\]

Cf. Ross to show that distribution is constrained

Suppose that \(N(t) = k = 5\) \(5 < t\), \(\lambda\) fixed

\[\mathbb{P}(N(s) = 3 \mid N(t) = 5)\]

\[= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(N(s) = 3, N(t) = 5)}{\mathbb{P}(N(t) = 5)} = \frac{\mathbb{P}(N(s) = 3) \mathbb{P}(N(t) - N(s) = 2)}{\mathbb{P}(N(s) = 5)}\]

\[\text{independent \(\Rightarrow\)} \frac{\mathbb{P}(N(s) = 3)}{\mathbb{P}(N(s) = 5)} \frac{\mathbb{P}(N(t) - N(s) = 2)}{\mathbb{P}(N(s) = 5)}\]

\[\therefore \mathbb{P}(\lambda \varepsilon) \sim N(t) \quad \text{expand to get} \quad \frac{s^5}{5!} \left( \frac{s}{t} \right)^3 \left( \frac{t-s}{t} \right)^2 ?\]
Get a binomial distribution with
5 trials
3 successes
probability of succeeding \( \frac{5}{t} \)
i.e., \( X \sim \mathcal{B}(n_2, \frac{5}{t}) \)

\[ P[N(s) = n_1, \mid N(t) = n_2] = P(X = n_1) \]

look up first 7 or 8 in section on Poisson process
Prove $Y_1 \sim \text{Exp} (\lambda + \lambda_2)$

where $Y_1 = \min (X_1, X_2)$ for $X_1 \sim \text{Exp} (\lambda_1)$
$X_2 \sim \text{Exp} (\lambda_2)$

independent

Find $Y_2 - Y_1$, show independently distributed $Y_1$

where $Y_2 \sim \max (X_1, X_2)$