Claim. The functor $\text{Res}_{[G/N]}^{[G]} : \mathbb{K}[G/N]-\text{Mod} \longrightarrow \mathbb{K}[G]-\text{Mod}$

is left adjoint to $(\cdot)^N$.

Proof. In view of 6.12(a), ETS $(\cdot)^N \cong \text{Coind}_{[G]}^{[G]}$ NTS. for $A$ in

$\mathbb{K}[G]-\text{Mod}$, $\exists$ nch hom. of $\mathbb{K}[G]$-modules

$A^N \cong \text{Coind}_{[G]}^{[G]}(A) = \text{Hom}_{\mathbb{K}[G]}^{\text{aff}}(\mathbb{K}[G/N], A)$

we know that for $N = G$

$$f \longmapsto f$$

check $(gf)(1) = f(1 \cdot g) = f(gN) = g(f(1))$ $g \in G$ \checkmark

As a consequence, we obtain an adjoint of functor

$$\text{Hom}_{\mathbb{K}[G]}(\text{Res}_{[G/N]}^{[G]}, -) \cong \text{Hom}_{\mathbb{K}[G/N]}(-, I^N) : \mathbb{K}[G]-\text{Mod} \longrightarrow \mathbb{K}[G/N]-\text{Mod}$$

The functor on the left is exact being the composite $\text{Hom}(-, I) \cdot \text{Res}$, both of which are exact, hence the functor on the right is exact.

Therefore

$$H^n(G/N, I^N) = \text{H}^n_{[G]}(\text{Hom}_{[G/N]}(P, I^N))$$

$P$ a q-

res 0

of $\mathbb{K}[G/N]$

(contr of $f - n > 0$)
Remarks

- H/S give two proofs of Milnor-Serre (1953)
  
  #1 using the Cartan-Eilenberg spec,
  
  #2 explicit construction with explicit filtrations of complexes
  
  This method gives “description” of the edge maps

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the weaker def’ of $\mathcal{S}$-functors, not requiring exactness.

- An application of LH555: $H^\ast(G,IK)$ is a graded commutative
  $\mathbb{K}$-algebra, and $H^\ast(G,IM)$ will $M$ a $\mathbb{K}$-module,
  this is a module
  
  over $H^\ast(G,IK)$, both via “cup products”

Theorem (Evens, 1961) G a finite group, $IK$ noetherian

i) $H^\ast(G,IK)$ is a finitely generated (“free”) $\mathbb{K}$-algebra (these noetherian)

ii) if $M_\mathbb{K}$ is noetherian then $H^\ast(G,IM)$ is a noetherian $H^\ast(G,IK)$-module

As many invariants arise from this.

Open problem: What about “finite” Hopf algebra, finite tensor categories

Another application of LH555

Lemma 2.5 $\mathbb{K} = \mathbb{C}^0$, free abelian of rank $r$, and $IK$ any commutative ring.

Then: $H^\ast(G,IK) \cong \mathbb{K}^r$ as $\mathbb{K}$-module of rank $\binom{r}{n}$ as $\mathbb{K}$-modules.

This can be initiated as $H^\ast(G,IK) \cong \vee (IK^n)$. This can also be deduced

from the Hochschild-Kostant-Remmert theorem.

Proof: Induct on $r$.

- $r = 1$: $H^0(C_0,IK) = IK$ (invariants) $= IK$
  
  $H^1(C_0,IK) = IK_{C_0}$ (co-invariants) $= IK$
Choose $N \leq C_\infty \leq G$ with $G/N \cong C_\infty$.

and look at LHS:

$$E^{p,q}_2 = H^p(G/N, H^q(N, k)) \Rightarrow H^*(G, k)$$

$\Rightarrow C_\infty$

Have $E^{p,q}_2 = 0$ if $q \neq 0, 1$ (from $C_\infty$-cohomology)

$$E^{p,q}_2 \Rightarrow H^*(G, k) \Rightarrow E^{p,q}_\infty$$

so $E^{p,q}_\infty = E^{p,q}_2$ stabilizes.

Get guess. Top slice of filtration

$$0 \rightarrow E^{p+1}_{\infty} \rightarrow H^*(G, k) \rightarrow E^{p,q}_2 \rightarrow 0$$

Moreover, the $G/N$ action on $H^*(N, k)$ is trivial, see Lemma 2.6 below.

Thus,

$E^{0,0}_2 = H^0(G/N, H^0(N, k)) = H^0(N, k) \cong k^{(r-1)}$

$E^{0,1}_2 = H^1(G/N, H^1(N, k)) = H^1(N, k) \cong k$

Induction $\Rightarrow$

$E^{1,0}_2 = H^1(G/N, H^0(N, k)) = H^0(N, k) \cong k$

Now $\otimes (\text{which splits})$ gives

$$H^*(G, k) \cong k^{(r-1)} \oplus k^{(r-1)} = k^{(r)}$$

In particular, we obtain cohomological dimension

$$\text{cd}_{ik}(C_\infty) = r$$

from

$$\text{cd}_{ik}(G) \leq \text{cd}_{ik}(GN) + \text{cd}_{ik}(N)$$

(earlier) plus induction

one obtains

$$\text{cd}_{ik}(N) \leq r$$

Since $H^*(C_\infty, k) \cong k$, we obtain.
Lemma 26:

Let \( N \subseteq Z(G) = \text{center of } G \)

Then for any \( M \) in \( l_k G \text{-Mod} \) and any \( g \in G \), the action of \( g \) on \( H^*(N, M) \) is given by the map

\[
H^*(N, M) \rightarrow H^*(N, M) \quad \text{where} \quad \mu_g : M \rightarrow M \quad m \mapsto gm
\]

(\( \mu \) is an isomorphism in \( l_k G \text{-Mod} \)).

In particular, if \( \mu_g = 1_M \) then \( g \) acts trivially on \( H^*(N, M) \).

Recall the explicit description of the \( G \)-action on \( H^*(N, M) \):

Choose a projective \( P \) of \( l_k G \); get a projective \( P^* \) of \( l_k N^\text{op} \). Then \( G \) acts on the cochain complex

\[
\text{Hom}_{l_k N} (P^*, M) \]

as follows:

\[
(\phi f)(p) := \phi f(g^{-1} p)
\]

\((\phi \in G, \phi \in P)\),

Needs: friendly \( f \) in \( l_k N \text{-lin} \), but \( G \) may act nontrivially.

One checks

\[
d^*(g f) = g (d^f)
\]

Thus

\[
d^* \text{ is } l_k G \text{-lin}
\]

Now assume \( N \subseteq Z(G) \). Then:

\[
(, ) \mu : \text{Hom}_{l_k N} (P^*, M) \rightarrow \text{Hom}_{l_k N} (P, M) \rightarrow \text{Hom}_{l_k N} (P, M)
\]

Want on \( H^*(N, M) \), the action \((, ) \mu \) induces \( H^0(N, M) \).

Next \( \mu_{g^{-1}} : P \rightarrow P \) is a map of projective \( l_k N \text{-lin} \) in \( l_k N \text{-Mod} \)

That lifts \( \mu_{g^{-1}} = 1 : P \rightarrow P \). Another lift is \( 1_P \), both homotopic by comparison then
So $\mu_{g-1} \sim 1_P$, and hence $\mu^*_{g-1} \sim 1_{\text{minim}(P,N)}$.

Thus in $H^0(N,\mu)$, $\mu^*_{g-1}$ gives $1$. \( \Box \)