Recall: If Prop 18

If $k$ a commutative

$G$ a group

diagonal action

$M, N \in \text{IkGMod}$

$\triangleright M \otimes N \in \text{IkGMod}$

$g(m \otimes n) = gm \otimes gn$

Prop 18 to do

$\text{pd}_{\text{IkG}} k \geq \text{pd}_{\text{IkG}} A$ for any $A \in \text{IkGMod}$

In fact, if $P$ is a projective of $k$ then $P \otimes A$ is a projective if $A \otimes k \equiv k \otimes A$. It remains to show that $\text{pd}_{P} A$

are in fact projective:

$P, A \in \text{IkGMod}$, with $P$ projective, then $P \otimes A$ is a projective $\text{IkG}$-module

To prove this, $\text{Ind}_{k}^{\text{IkG}} A \otimes k$ is projective as $\text{IkG}$-module

Now $\text{Ind}_{k}^{\text{IkG}} (A \otimes k) = (k \otimes A \otimes k)$ is $\text{IkG}$-action on left

$A$ is projective in fact since $k$ is a field.

The only $A$ the fact that tensor products respect direct sums.

Nevertheless $\text{Ind}_{k}^{\text{IkG}} A \otimes A$ is projective due to $\text{Ind}_{k}^{\text{IkG}} A$

left action

diagonal action

This arises as follows. Take a $k$-linear map

$\phi: A \rightarrow k \otimes A$

By Frobenius Reciprocity Lemma 6 this lifts uniquely to a $\text{IkG}$-linear map.
\[ \text{Ind}_k(A/k) \rightarrow kG \otimes A \]

\[ \alpha \otimes a = \alpha(1 \otimes a) \]

\[ (\alpha \otimes kG) \alpha = \sum k_g g \]

\[ \alpha(1 \otimes a) = \sum k_g (g \otimes g a) \]

so, e.g.

\[ g \otimes a \rightarrow g \otimes g a \]

This map is easily seen to be an isomorphism.

\[ \S 3 \] Construction of Spectral Sequences

Exact couples & filtrations (not deeply explained in Weibel)

Will work in \( R \)-Mod for simplicity.

(1) From exact sequences to spectral sequences (Massey)

\[ \text{Def} \quad \text{An EXACT COUPLE is a pair of bigraded modules} \]

(\( \text{with two degrees} \)) \[ D = \bigoplus D_{p,q} \quad \text{and} \quad E = \bigoplus E_{p,q} \]

(\( \text{direct sums of} \) \( R \)-modules)

together with \( R \)-module maps \( \alpha, \beta, \gamma \) each of some bidegree such that

the following triangle is exact at each vertex, \( \text{im} \alpha = \ker \beta \)

\[ \begin{array}{ccc}
D & \xrightarrow{\alpha} & D \\
\downarrow \gamma & & \downarrow \beta \\
E & \xrightarrow{\beta} & F
\end{array} \]

Suppose one \( D, E, \alpha, \beta, \gamma \) given. Then a differential

\[ d = \beta \gamma \]

To get \( d : E \rightarrow E \) with \( d^2 = \beta \gamma \beta \gamma = 0 \)
We get another exact couple, the derived couple

\[ D' = \alpha(D) \xrightarrow{\alpha'} \alpha'(D) = D' \]

With \( D' \) and \( E' \) inheriting the grading from \( D \) and \( E \).

For example,

\[ (E')^p = (\ker d) \cap E^p \]

and the component of \( E' \) comes from a component of \( E \).

"Bigraded"

\[ f: \bigoplus M^p \rightarrow \bigoplus N^p \]

has biproducts, \((r,s)\), means

\[ f(M^p, r) \subseteq N^{p+r, q+s} \]

and the following maps:

Recall

\[ D \xrightarrow{\beta} D \]

\[ d = B \delta \]

\[ \alpha' = \alpha|_{D'} \]

\[ \delta' = \delta'([e]) = \delta(\alpha) \quad e \in \ker |d| \]

\[ \beta' = \beta'(\alpha) : = \beta(\delta) \quad J \in D \]

This is all well-defined, for example

\[ 0 \in \ker |d| \Rightarrow \beta'(\alpha) = 0 \quad \text{Conclude} \quad \delta \]
Introducing this process, we obtained the derived couple with $r = 1$
being the original derived couple $(D_r, E_r, \alpha_r, \beta_r, \delta_r)$

\begin{align*}
(D_r, E_r, \alpha_r, \beta_r, \delta_r)
\end{align*}

Recall

\begin{align*}
E_{r+1} = H(E_r, f_r = \beta_r \delta_r)
\end{align*}

Remark: a spectral sequence is a family of modules with differential $d_r$ of degree $1 - r$, and other degrees.

**Lemma 10**

Let $(D_r, E_r, \alpha_r, \beta_r)$ be an exact couple with the following bidegrees

- $\text{bideg } \alpha = (-1, 1)$
- $\text{bideg } \beta = (0, 0)$
- $\text{bideg } \delta = (1, 0)$

Then $\text{bideg } \alpha_r = (-1, 1)$, $\text{bideg } \beta_r = (r-1, 1-r)$ \text{bideg } \delta_r = (1, 0)

in particular $\text{bideg } d_r = (r, 1-r)$ \text{(as required in §1)}

**Proof** Induct on $r$:

Induction on $r$:

\begin{align*}
X_{r+1} = \left[ \alpha_r \right] \text{ same degree as } \delta_r \text{ because just restricted and modified}
\end{align*}

\begin{align*}
\alpha_{r+1} = \alpha_r |_{\left[ \alpha_r(D_r) \right]} \text{ same bidegree}
\end{align*}

\begin{align*}
\beta_{r+1} = \delta_r |_{\left[ \beta_r(d_r) \right]} \text{ bidegree}
\end{align*}

For $\beta_{r+1}$\quad $\alpha_r(d) \in D_r^{r,n} = \alpha_r(D_r)$

\begin{align*}
\beta_{r+1}
\end{align*}

\begin{align*}
\left[ \beta_r(d) \right] \in D_r
\end{align*}

By induction $\beta_r(d) \in D_r^{r-1, q-1} = D_r^{r-1, q-1}$
where \( \beta_{r+1} = (r, -r) \) as desired.

Thus \( \{E_r, d_r\} \) is a spectral sequence (cohomology).

(2) The exact couple of a filtered complex

**Definition.** Let \( C = (C^i, d^i) \) be a cochain complex of \( R \)-modules.

A decreasing filtration of \( C \) is a family of subcomplexes

\[ F^pC \]

with \( F^pC \supseteq F^{p+1}C : \forall p \in \mathbb{Z} \)

and \( \bigcap \mathbb{N} \) \( R \)-submodules \( C' \) with \( F^pC \supseteq F^{p+1}C' \) and \( d(F^pC') \subseteq F^{p+1}C' \)

**Lemma 2.**

A decreasing filtration \( \{F^pC\}_{p \in \mathbb{Z}} \) of \( C = (C^i, d^i) \) determines an exact couple \( (D, E, \alpha, \beta, \gamma) \) with the bidegrees as in (1.17):

\[ \alpha: (1, -1), \quad \beta: (0, -1), \quad \gamma: (1, 0) \]

The \( r \)-th derived complex has

\[ \frac{\partial_r}{\partial_r} \bigoplus_{q \geq r} F^{-q} \rightarrow H^q(F^p) \rightarrow H^{q+1}(F^p) \]

If \( F^p = F^pC' \). For each \( p \), have a set of cochain complexes

\[ 0 \rightarrow F^pC' \rightarrow F^pC' \rightarrow F^pC' \rightarrow 0 \]

so get the long cohomology sequence.

\[ \ldots \rightarrow H^n(F^{p+1}) \rightarrow H^n(F^p) \rightarrow H^n(F^p/F^{p+1}) \rightarrow H^{n+1}(F^p) \rightarrow \ldots \]
Put \( n = p + q \) and
\[
D^{p,q} := H^{p+q}(F^p) \quad \text{and} \quad E^{p,q} = H^{p+q}(F^p/F^{p+1})
\]
to rewrite the long cohomology sequence as
\[
\cdots \longrightarrow D^{p+1,q-1} \overset{d}{\longrightarrow} D^{p,q} \overset{d}{\longrightarrow} E^{p,q} \overset{d}{\longrightarrow} D^{p,q+1} \longrightarrow \cdots.
\]
This gives the desired exact couple. The last assertion is also clear.

\( \square \)