(2,1) - graphs which 'develop to (2,3) graphs'

finite directed graphs colored by group elements

More on Random Graphs

Chromatic Number vs Girth

Recall: The chromatic number of $G$, $\chi(G)$, is minimal $k$ s.t. we can assign each vertex one of $k$ colors s.t. no edge has monochromatic endpoints.

Fact: computing $\chi(G)$ is NP complete. Thus don't expect much 'local structure' in the problem.

These ideas appear in 1978 or so.

1958 (Erdős) There is a $k > 0$ Edge graph of girth $\geq k$, chromatic number $\geq k$.

Recall, the girth is the length of the shortest cycle.

Proof sketch:

* uses probabilistic method. Define a distribution over graphs, show $P_1(\chi(G) > k \& \text{girth}(G) > k)$ is bigger than zero, infer that it exists.

Use random graphs $G(n, p)$ but no $p$ s.t. * holds. Exercise

Need to use alteration scheme.
**Lemma** If $\chi(G)$ is independent number (size of biggest independent set)

$\chi(G) \geq \chi(G)$

If Colors are independent

Contr. $\chi(G) > \chi(G)$

Pick $0 < \lambda < \frac{1}{6}$ and $p = \frac{\lambda - 1}{n}$

Bound expected number of short cycles in $G(n, p)$

$$\text{probability of seeing cycle length i} = \binom{n}{i} p^i \leq \frac{n^i p^i}{i!}$$

$$E(\text{# short cycles}) = \sum_{i=3}^\infty n^i p^i = \sum_{i=3}^\infty n^{i \lambda} \leq \ln n^{1/\lambda} = O(n) \quad \text{and} \quad \lambda > 1$$

For $n$ large, this is $< \frac{n^n}{4}$

By Markov's $\leq$, for $n$ large $Pr(\geq n^{1/2} \text{ short cycles}) < \frac{1}{2}$

Set $a = \frac{6}{p} \log n$. Want to estimate $Pr(\chi(G(n, p)) > a)$

If this is small ($< \frac{1}{2}$), then have w.p positive probability, $G(n, p)$ has $< \frac{n}{2}$ short cycles and no independent sets either.

⇒ "Alteration" throw out $\geq a^2$ vertices to kill cycles but not increase

1. $\geq$ spanner independent set past bound.

Expected # of independent size $\geq a$ is $\leq \binom{n}{a}(1 - p)^{\binom{a}{2}} \leq n^a e^{-p(a^2)} \to 0$

and $1 + x \leq e^x$

For $n$ large $\chi(G(n, p))$ is with high probability < $a$

Now $p = \frac{\lambda - 1}{n}$ so $\lambda = \frac{6}{\log n} \log n \to \infty 
$

So we can make $\chi(G)$ small, and thus $\chi(G)$ large.