Graphs
22 February 2011

Circuits \(\subseteq\) Bases
for circuits, define
\[ B = \{ B \in E : B \text{ maximal set w/o circuits} \} \]

Check (3)
\[ B_1, B_2 \in B \quad (as \text{ def.1}) \]

Look at \( B_1 + e_2 \quad e_2 = B_2 - B_1 \), suppose
\( B_1 + e_1 \) has > 1 circuit
\[ e_1 \in C_1 \cap C_2 \]
\[ \Rightarrow (3) (C_1 \cup C_2) \setminus e \supseteq C_3 \in C \]
\[ C_3 \subseteq B \quad \text{contradicts} \quad B_1 \in B \]

Challenge rank function thing.

Planar Graphs

Def A graph \( G \) is PLANAR if "we can draw it in the plane," that is, if we can
- Assign a point \( p_i \) to each vertex \( i \) (distinct)
- Assign a continuous curve \( C_{ij} \) to each edge \( \{i,j\} \),
  \( s.t. \ C_{ij}(0) = p_i \quad C_{ij}(1) = p_j \quad \) and
  distinct \( C_{ij} \) \( C_{kl} \) do not intersect other than endpoints
- \( C_{ij} \) does not intersect \( p_k \) for \( k \neq i,j \)

Def Such a construction is a \textit{drawing} of a planar graph.
Ex Trees, cycles are planar
   hint - grow trees with an expanding circle

Def Given a drawing of planar graph $G$, the
   faces of $G$ are the connected sets (faces) of the
   complement of the drawing in the plane
   combinatorially, we identify
   faces with vertex induced
   simple cycles

Notice - faces are induced cycles

× Jordan curve theorem — $C$ continuous closed
   simple curve (simple - embedding of circle into plane - no
   self-intersections, $S \rightarrow \mathbb{R}$ continuous)

Then $\mathbb{R}^2 \setminus C$ has two connected components
   "inside" and
   "outside"

Note, we count the "outer face"

spherical projection,
the face which contains north
pole is outer face
For graphs, \( 2m = \sum_{v \in V} \text{deg}(v) \)

how many faces is each edge in? \( \& \)-edges of

if \( G \) is 2-connected, by Jordan curve theorem

\[ 2m = \sum_{f \in \text{faces}} |f| \]  
where |f| = # vertices in \( f \)

If \( G \) is planar but not 2-connected, it is connected a bunch of planar graphs connected by cut vertices

\[ \triangle \triangle \rightarrow \triangle \triangle \triangle \]

octahedron

**Def A planar dual of \( G \), \( G^\ast \)**

\[ V(G^\ast) = F(G) \]

\[ F(G^\ast) = V(F) \]

\[ E(G^\ast) = E(G) \]
Observation — the 2nd counting relation is dual to the first observation — In matroidal terms

The co-circuits of the tree matroid are cuts.

For planar graphs, the cuts are cycles in the planar dual.

So if \( G \) is planar, the dual tree matroid \( \cong \) tree matroid of the dual.

Proof Preview — this is reversible!

**Thm.** Let \( G \) be planar with

\[ |V| = v, \quad |E| = e, \quad |F| = f. \]

(Euler's formula)

\[ v - e + f = 2. \]

**Pf.** Key claim: Let \( T \) be an acyclic subgraph of \( G \). Then \( T^* \), the dual of \( E \setminus T \), is connected, so \( T^* \cong (E \setminus T)^* \).

Apply claim to spanning tree \( T \) of \( G \). Then

- \( T^* \) connected, \( (E) T = T^* \) so \((E) T^* \) is acyclic
- so \( T^* \) is a spanning tree of \( G^* \)

\((v - 1) + (f - 1) = e \) (i.e., edges of \( T, T^* \) cover edges of \( G \))
But the key claim follows from the fact that $G$ planar $\Rightarrow$ dual tree matroid $\cong$ tree matroid of dual $G'$

because

$G'$ subgraph $G'$ disconnects $(G')^\ast$ $\iff$ it has a cycle

Corollaries (Exercises)

1) Any planar graph has $O(\sqrt{V})$ edges

2) $K_5$ not planar

3) $K_{3,3}$ is not planar

4) if planar, faces are all triangles

4) Average degree of planar graph $< 6$