Last time: characterize trees by matchings

Recall \( \mathcal{T} \) is a tree \( \iff \mathcal{T} \) has \( n - 1 \) edges.

For subgraphs \( \mathcal{G} \) with \( m' \) edges and \( n' \) vertices, we have \( m' \leq n' - 1 \).

Proof sketch: any subgraph \( \mathcal{G} \) with avg degree \( 2 \) has a cycle.

Recall \( (2) \) from last time.

Used following trick:
\( \mathcal{G} = (V, E) \) defined \( B_{\mathcal{G}}(\mathcal{G}) = (V^k, E, F) \).

\[ \text{(k copies of } V \text{)} \]

Showed using Hall's Theorem.

For all subgraphs of \( \mathcal{G} \), \( m' \leq kn' \).

\( \iff \)

\( B_{\mathcal{G}}(\mathcal{G}) \) has a complete matching.

Proof Hall's.
Ex

See if we can reduce # of modules needed to check

Associate with thick - thin problem with thin - thick versions of orientations

Ex

Reinterpolate all thin - thin versions of orientations if thin

Ex
This time: Graphs that satisfy some hereditary condition.

Idea: Generalize some properties of trees

**Def** Let \( G \) be a graph. \(|V| = n, |E| = m\)

\( G \) is \((k,l)\)-sparse if all subgraphs \( G' \) we have

\[ m' \leq kn' - l \]

\( G \) is "tight" or a \((k,l)\)-graph if in addition

\[ m = kn - l \quad \text{i.e. at top level} \]

**Eg.** Trees are "\((1,1)\)-graphs"

Forrests are \((1,1)\)-sparse

**Eg.** \((1,0)\) graphs have exactly one cycle per connected component

**Eg.**

A \((k,l)\) graph has average degree

\[ \frac{2m}{n} = \frac{2(kn - l)}{n} \leq pk \]

Turan's thm: independent set

3 independent set of size \[ \frac{n}{dk} \]

Conclusion: if \( k \) doesn't depend on \( n \), no \((k,l)\) graph can have \[ m' = kn' - l \] for all subgraphs.
Ex

All bicycles are subdivisions of either:

- subdivision

these are all, and all of, the minimal not (1,1) sparse graphs.

Eg. If cycles minimal bad for (1,1) graphs

- G minimal not (1,1) - sparse graph
- G connected, or else discarded one of components is (1,1)-free
- If G has a leaf, m \geq n, remove leaf m=1 \geq n-1
- so G remove leaves is still not (1,1) sparse
- If there is more than one cycle, exists an edge we can remove to maintain the existence of a cycle
If in this case \( m > n \), because if exactly one cycle, \( m = n \). But now if we remove any edge, \( m - 1 > n \), so a cycle remains.

If 2 cycles share a path, there exists a third cycle by pasting together the outside.

Interesting fact:

Cycles are specific examples of \((1,0)\) graphs.

Notice further: if we add an edge to a tree, we get a connected \((1,0)\) graph.

Question: for what \( k, l > 0 \) do we have a nonempty class of \((k,l)\) graphs?

Fix \( k, l \in \mathbb{N} \)

Try \((1,2)\) graphs

None exist, because \( m' = 1 \) for one edge \( n' = 2 \).

=> Any \((1,2)\) sparse has no edges at all.

Thm (Existence of \((k,l)\) graphs)

Let \( k, l \) be natural \( \in \mathbb{N} \). Then an \((k,l)\) graph if

\( \star \) \( l < 2k \)

or \( n \) large enough that \( \binom{n}{2} \geq kn - l \).
If and \( n_0 \) be maximum s.t. \( K_{n_0} \) is \((k, l)\)-sparse.

Add one vertex

\[
K_{n_0}
\]

add edges to \( H \) until \( m = kn - l \)

(works since \( K_{n_0} \) not sparse)

This will be a \((k, l)\)-graph on \( n_0 + 1 \) vertices to check. If \( G \) subgraph \( G \)

\[
G \subseteq K_{n_0}
\]

were done

if not then \( \forall (u, v) \in E \) same \( n_0 + 1 \) vertices, four edges.

To finish, for larger \( n \). If

\[
G_n
\]

is a \((k, l)\)-graph on \( n \) vertices

then we can make \( G_{n+1} \) obtained by

\[
G_{n+1}
\]

same argument says \( G_{n+1} \) is a \((k, l)\)-graph.

Thm (Devin, Reiner) This is everything interesting,

i.e. non-integer combination don't make \((k, l)\)
**Def** Let $G$ be $(k, l)$ sparse.
A subgraph $G'$ is a **block** if it is itself a smaller $(k, l)$ graph.

**Def** A subgraph $G'$ is a **component** if it is a maximal block.

n.b. $(1, 1)$ components of a $(1, 1)$ sparse graph are the connected components.

**Thm** (Structure Theorem)

Let $G$ be $(k, l)$ sparse and let $G', G''$ be blocks in $G$.
- $G', G''$ are vertex induced.
- $G' \cap G''$ (vertex induced intersection) is a block.
  - If $l = 0$, $G' \cup G''$ is a block.
  
**In general**, $G' \cup G''$.
- If $l \leq k$ and $G' \cap G'' \neq \emptyset$ then $G' \cup G''$ is also a block.
- If $l > k$ and $|G' \cap G''| > 1$ then $G' \cup G''$ is also a block.