Extension of Functions in $W^{1,p}(\Omega)$ \(1<p<\infty\)

**Thin** \(\Omega\) ball, \(\partial\Omega\) is \(C^1\). Then \(\exists E:W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)\)

* \(E\) is linear
* \(Eu = u\) a.e. in \(x \in \Omega\)
* \(\exists C>0\) indep. of \(u\) s.t.
  \[\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}\]

**Proof** local estimations

* partitions of unity to extend \(u\).

1st assumption - \(\partial \Omega\) is flat near \(x_0\), i.e.

is a section of hyperplane or line.

So \(\exists \Omega \cap B(x_0, r) = \{(x_1, \ldots, x_n) : x_n = 0\}\)

for any rotation.

\[B^+ = B_r(x_0) \cap \{x_n > 0\}\]

\[B^- = B_r(x_0) \cap \{x_n < 0\}\]

will extend into \(B^-\)

need to straighten boundary
\[ u \in C^\infty(\overline{\Omega}) \quad C^1(\overline{\Omega}) \]

Define
\[
\overline{u}(x) = \begin{cases} 
  u(x) & x \in \overline{B}^+ \\
  -3u(x_1, \ldots, x_{n-1}, -x_n) & x \in \overline{B}^- \\
  + \frac{1}{4} u(x_1, \ldots, x_n, -\frac{x_n}{2}) & 
\end{cases}
\]

Claim
\[ \overline{u}(x) \in C^\infty(\mathbb{B} \setminus \{x_n = 0\}) \text{ i.e. except on boundary} \]

All partials except \( x_n \) finite on boundary.

\[ \frac{\partial \overline{u}}{\partial x_n}(x) = \frac{\partial u}{\partial x_n}(x) \]

\[ = 3 \frac{\partial u}{\partial x_n}(x_1, \ldots, -x_n) + \frac{1}{2} \frac{\partial u}{\partial x_n}(x_1, \ldots, \frac{x_n}{2}) \]

\( x_n \to 0^+ \quad u(x) \in \overline{B}^+ \) so \( \overline{u}(x) \to 0^+ \)

\( x_n \to 0^- \)

\[ \frac{\partial \overline{u}}{\partial x_n} = 3 \frac{\partial u}{\partial x_n}(x_1, \ldots) - 2 \frac{\partial u}{\partial x_n}(x_1, \ldots, \frac{x_n}{2}) \to \frac{\partial \overline{u}}{\partial x_n}(x', 0) \]

Now
\[ \frac{\partial \overline{u}}{\partial x_n}(x', 0) \]

\[ D^\alpha \overline{u}(x) = \begin{cases} 
  D^\alpha u(x) & x \in \overline{B}^+ \\
  -3 D^\alpha u(x'-x_n)(-1)^{\alpha_n} + 4 D^\alpha u(x, -\frac{x_n}{2})(-1)^{\alpha_n} & 
\end{cases} \]

\( W^{1,p} \) so need only order 1 so don't need the above.
Claim

\[ \| \overline{u} \|_{W^{1,p}(B)} \leq C \| u \|_{W^{1,p}(B^+)} \]

\[ = \int_{B} |u(x)|^p \, dx + \int_{B^+} \left| -3u(x, y_n) + 4u(x, \frac{y_n}{2}) \right|^p \, dx \, dy \]

\[ \leq \left( \int_{B} |u(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{B^+} \left| -3u(x, y_n) + 4u(x, \frac{y_n}{2}) \right|^p \, dx \, dy \right)^{\frac{1}{p}} \]

\[ =: A + B \]

\[ B \leq \left( \int_{B^+} 3u(x, y)^p \, dx \, dy \right) \left( \int_{B^+} 4u(x, \frac{y_n}{2}) \, dx \, dy \right)^{\frac{1}{p}} \]

\[ \leq A \left( \int_{B} u \, dx \right)^{\frac{1}{p}} + 4 \left( \int_{B_{1/2}^+} u(x, y_n)^p \, dx \, dy \right)^{\frac{1}{p}} \]

where \( B_{1/2}^+ \) has cut \( y_n \) in half.

\[ \| \overline{u} \|_{L^p(B)} \leq (3 + 4.2^{\frac{1}{p}}) \| u \|_{L^p(B^+)} \]

\[ \| D^a \overline{u} \|_{L^p(B)} \leq C_p \| D^a u \|_{L^p(B^+)} \]
Now need to straighten out the boundary.

Now, $\Omega$ is not necessarily flat.

$\Omega \cap B_r(x_0) = \{ (x', x_n) : x_n = \Phi(x') \}$

that is, after rotation $x_n$ is a $C^1$ function of the other coordinates.

$\Omega \cap B_r(x_0) = \{ (x', x_n) : x_n > \Phi(x') \}$

$\Phi(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, x_n - \Phi(x_1, \ldots, x_{n-1}))$

$\Phi: \Omega \cap B_r(x_0) \to V^+$

notice on the boundary, $\Phi$ is 0

So $V^+$ is like

**Jacobiarn**

$J_{\Phi}(x_1, \ldots, x_n, x_n) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{vmatrix}$

$\det J_{\Phi} = 1$

by inverse function then, are neighborhoods near $x_0$, $\Phi(x_0)$
\[ \exists \ V^+ \text{ neighborhood of } \phi(x_0) \text{ and } r \ \text{ s.t.} \]

\[ \tilde{\phi} : \Omega \cap B(x_0, r) \to V^+ \]

\[ \tilde{\phi} \text{ is } 1-1 \text{ and onto.} \]

Let \( \psi : V^+ \to \Omega \cap B(x_0, r) \) is \( C'(V^+) \) since inverse of \( C' \) function.

Let \( u'(y) = u(\psi(y)) \) \( y \in V^+ \).

Let \( \bar{u}' \) be the extension of \( u' \) to \( V^+ \cup V^- \).

Thus

\[ \| \bar{u}' \|_{W^{1,p}(\mathbb{R}^n)} \leq C \| u' \|_{W^{1,p}(\mathbb{R}^n)} \]

\[ \| \bar{u}' \|_{W^{1,p}(V^-)} \leq C \| u' \|_{W^{1,p}(V^+)} \]

\( \tilde{\phi} \) was defined above for differentiability.
\[ \| \bar{u} \|_{L^p(V)} = \left( \int_{V^+} |\bar{u}'(x)|^p \, dx + \int_{V^-} |\bar{u}'(x)|^p \, dx \right)^{\frac{1}{p}} \]

\[ = \left( \int_{V^+} |\bar{u}'(x)|^p \, dx + C \int_{V^+} |u'(x)|^p \, dx \right)^{\frac{1}{p}} \]

\[ \leq C \int_{V^+} |u'(x)|^p \, dx \]

Same argument for the gradient.

Need to return to \( u \)

\[ \| \bar{u}' \|_{W^{1,p}(V)} \leq \| u \|_{W^{1,p}(U)} \]

\[ \int_V |\bar{u}(x)|^p \, dx = \int_{\Omega \cap B} |\bar{u}'(\phi(y))|^p |J\phi(y)| \, dy \]  

\[ (x = \phi(y)) \]

\[ \rightarrow_{\rho} \]

\[ = \int_V |u(y)|^p \, dy \]

\[ \Rightarrow \| \bar{u} \|_{W^{1,p}(V)} \leq C \| u \|_{W^{1,p}(U)} \]

\[ V = U^+ \cup U^- \] so extended into a neighborhood of \( \bar{x}_0 \)
Let $x_0, V_{x_0}, U_{x_0}$ so that is true.

$\partial \Omega$ compact, so finite covering by $V_1 \ldots V_N$ and

$$
\| \overline{u_i} \|_{W^{1,p}(V_i)} \leq C \| u \|_{W^{1,p}(U_i)}
$$

There is a $V_0 \subset \Omega$ st.

$V_0 \subset \Omega$ st. $V_0 \cup V_1 \cup \ldots \cup V_N \supset \overline{\Omega}$

Take a partition of unity

Let $\phi_0, \phi_1, \ldots, \phi_N$ be partition of unity

subordinated to $V_0 \ldots V_N$. Let $\overline{f(x)}$

$$
\overline{u(x)} = \sum \overline{u_i} \phi_i
$$

$\overline{u(x)} \in C^1(\mathbb{R}^n)$ since $\phi_i$ have compact support

$x \in \partial \Omega \rightarrow \overline{u}(x) = \sum_{i=0}^{N} \overline{u_i(x)} \phi_i(x) = u(x) \sum_{i=0}^{N} \phi_i(x)$

So $\overline{u} = u$ in $\Omega_0$.
\[ \| \overline{u} \|_{W^{1,p}(\mathbb{R}^n)} \leq \sum_{i=0}^{N} \| \overline{u}_i \phi_i \|_{W^{1,p}(\mathbb{R}^n)} \]

N.B., \( \text{supp}(\phi_i) \subset V_i \)

\[ = \sum_{i=0}^{N} \| \overline{u}_i \phi_i \|_{W^{1,p}(V_i)} \]

\[ \leq C \sum_{i=0}^{N} \| \overline{u}_i \|_{W^{1,p}(V_i)} \]

\[ \leq C' \sum_{i=0}^{N} \| u_i \|_{W^{1,p}(U_i')} \]

\[ \leq C'' \| u \|_{W^{1,p}(\Omega)} \]

\( \forall u \in C^1(\overline{\Omega}) \) we have \( Eu \in W^{1,p}(\mathbb{R}^n) \)

\( \text{Eu} = u \) in \( \Omega \) and \( \| \text{Eu} \|_{W^{1,p}(\mathbb{R}^n)} \leq C \| u \|_{W^{1,p}(\Omega)} \)

\( \text{NTS linear operation - Gut entire construction was linear. (The flip of} \ V' \text{was linear.} \)

\( \exists \Omega \in C^1, \) given \( u \in W^{1,p}(\Omega) \),

\[ \exists u_m \in C^\infty(\overline{\Omega}) \quad u_m \to u \text{ in } W^{1,p}(\Omega) \]

\( \{ E u_m \} \) is Cauchy so \( Eu := \lim_{m \to \infty} E u_m \)

\( * \) uniqueness from norm bound \( \leq \).