Wave Equation

\[ x \in \mathbb{R}^n, \quad t > 0 \]
\[ u = u(x, t) \]

\[ \square u = u_{tt} - c^2 \Delta_x u = 0 \]

\( c \) is constant
1) a string
2) waves in water, on a surface
3) Electromagnetism, acoustics, etc
Maxwell equations

Solve \( \square u = 0 \) in \( \mathbb{R}^n \times (0, +\infty) \)

Boundary conditions
\[ u(x, 0) = f(x) \quad \text{initial value position} \]
\[ u_t(x, 0) = g(x) \quad \text{initial velocity} \]

Important to solve 1st case when \( n = 1 \)
\[ u_{tt} - c^2 u_{xx} = 0 \quad x \in \mathbb{R}, \quad t > 0 \]

Write as composition of two first order equations
\[ \partial_{tt} - c^2 \partial_{xx} = (\partial_t + c \partial_x \partial_t - c \partial_x) \]
\[ \text{or} \quad = (\partial_t - c \partial_x \partial_t + c \partial_x) \]
Find $u$ such that

$$(\partial_t - c \partial_x) u = \nu$$

First solve

$$\begin{align*}
(\partial_t - c \partial_x) \nu &= 0 \\
(\partial_y - c \partial_x) \nu &= 0 \\
- c \partial_x \nu + \partial_y \nu &= 0
\end{align*}$$

$x(t,s) \quad y(t,s) \quad z(t,s) \quad (s,0, h(s))$

$$\begin{cases}
\dot{x} = -c \\
\dot{y} = 1 \\
\dot{z} = 0
\end{cases} \quad \begin{cases}
x(t,s) = -ct + s \\
y(t,s) = t + 0 \\
z(t,s) = h(s)
\end{cases}$$

Then

$$\nu(x,y) = z(x,y) = h(x+cy)$$

new solve $u$

$$\begin{align*}
\partial_t u + c \partial_x u &= \nu = h(x+ct) \\
\text{rename } t \text{ as } y
\end{align*}$$

$$\begin{align*}
(\partial_y + c \partial_x) u &= h(x+cy) \\
(x(t,s) \quad y(t,s) \quad z(t,s)) \quad &= (s,0, h(s))
\end{align*}$$

$$\begin{cases}
\dot{x} = c \\
\dot{y} = 1 \\
\dot{z} = h(x+cy)
\end{cases} \quad \begin{cases}
x(t,s) = ct + s \\
y(t,s) = t + 0 \\
z(t,s) = h( ct + s + ct) = h(s + 2ct)
\end{cases}$$

has solution $\nu(x,t) = h(x+ct)$ for any $h \in C^2$
\[ z(t, s) = \int_0^t h(s + \alpha \eta) \, d\eta + \omega(s) \]

\[ u(x, y) = z(y, x - cy) \]

From above

\[ = \int_0^y h(x - cy + \alpha \eta) \, d\eta + \omega(x - cy) \]

Switching \( t \) for \( y \) again

\[ u(x, t) = \int_0^t h(x - ct + \alpha \eta) \, d\eta + \omega(x - ct) \]

\( h \) is an arbitrary continuous function

\( \omega \) is an arbitrary continuous function \( \mathbb{C}^2 \) function

We wish to satisfy the initial conditions

\[ u(x, 0) = f(x) \]

\[ u(x, 0) = g(x) \]

\[ u(x, 0) = \omega(x - c0) = \omega(x) \]

So \( \omega(x) = f(x) \)

\[ \frac{\partial}{\partial t} \int_0^t \int_{-\alpha}^{\alpha} h(x + c(2\eta - t)) \, d\eta \, d\eta = \int_0^t h(x + cz) \, dz \]

\( \eta = \tau \rightarrow z = \tau t \)
\[ \frac{\partial}{\partial t} F(x,t) = \frac{1}{\alpha} h(x+ct) + \frac{1}{\alpha} h(x-ct) \]

So
\[ u(t,x) = \frac{1}{\alpha} h(x+ct) + \frac{1}{\alpha} h(x-ct) - c \omega'(x-ct) \]

So
\[ u(t,x) = h(x) - c \omega'(x) = g(x) \]

\[ \omega(x) = f(x) \]

\[ h(x) = g(x) + c f'(x) \]

\[ u(x,t) = f(x-ct) + \int_0^t g(x-ct+\alpha \eta) + c f'(x-ct+\alpha \eta) \, d\eta \]

\[ = f(x-ct) + \left[ \int_0^t g(x-c(\alpha \eta - t)) \, d\eta + c \int_0^t f(x-ct+\alpha \eta) \, d\eta \right] \]

\[ \therefore \frac{\partial}{\partial \eta} f(x-ct+\alpha \eta) = f'(x-ct+\alpha \eta), \quad 2c \]

\[ = f(x-ct) + \int_0^t g(x+c(2\eta - t)) \, d\eta + \frac{1}{2} \left[ f(x-ct+2c \eta) \right] - f(x-ct) \]

\[ u(x,t) = f(x-ct) + \frac{1}{\alpha} \left[ f(x+ct) - f(x-ct) \right] + \int_0^t g(x+c(2\eta - t)) \, d\eta \]

\[ x + c(2\eta - t) = \frac{3}{2} \]

\[ \therefore \int_0^t g(x+c(2\eta - t)) \, d\eta = \int_{x-ct}^{x+ct} g\left( \frac{3}{2} \right) \frac{d \sigma}{\alpha c} \]
\[
    u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{c} \int_{x-ct}^{x+ct} g(\xi) \, d\xi
\]

Theorem \( f \in C^2(\mathbb{R}) \) \( g \in C^1(\mathbb{R}) \) \( \Rightarrow \square u = 0 \) \( \text{in} \mathbb{R} \times (0,\infty) \)

\[
    u(x,0) = f(c)
\]

\[
    u_t(x,0) = g(c)
    \quad \text{(check!)}
\]

\( \text{[D'Alambert]} \)

\[
    y = \frac{ct}{c^2} (x-x_0) + t \quad y = \frac{x-x_0}{c} + t
\]

\[ y = \left( \frac{x-x_0}{c} \right) + c \]

and only \( f \) at endpoints \( x_0 - ct, \ x_0 + ct \)

\( g \) on interval in between.

With heat equation, solution has form

\[
    u(x,t) = \int \text{e}^{-\frac{(x-y)^2}{4t}} f(cy) \, dy
\]

\[
    u_t - \Delta_x u = 0
\]

whereas wave equation is hyperbolic.
METHOD of Poisson

Solution of \( \nabla U = u_{tt} - c^2 \Delta_x u = 0 \)

in \( \mathbb{R}^n \times (0, \infty) \)

Method due to Poisson, method of Spherical Averages

\( h(x) \) continuous in \( \mathbb{R}^n \)

\( r > 0 \)

\[ M_h(x, r) = \frac{1}{\omega_n r^{n-1}} \int_{|x-y|=r} h(y) \, dy \]

i.e. average on sphere radius \( r \) centered on \( x \).

where \( \omega_n \) surface area of unit sphere is \( \mathbb{R}^n \)

\[ \omega_n = \frac{n \pi^{n/2}}{\Gamma(n/2 + 1)} \]

but \( y = x + r \xi \quad d\sigma(y) = r^{n-1} \, d\sigma(\xi) \)

So

\[ M_h(x, r) = \frac{1}{\omega_n r^{n-1}} \int_{|\xi|=1} h(x + r \xi) \, d\sigma(\xi) \]

\[ = \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r \xi) \, d\sigma(\xi) \]

\( x \) fixed
\[ \partial_r M_h(x, r) = \frac{1}{\omega_n} \partial_r \left( \int_{|\zeta| = 1} h(x + r \zeta) \, d\sigma(\zeta) \right) \]

Assume \( h \) is \( C^2 \)

\[ = \frac{1}{\omega_n} \int \sum_{j=1}^{n} \frac{\partial h}{\partial x_j}(x + r \zeta) \cdot \zeta_j \, d\sigma(\zeta) \]

The idea

surface integral of derivative, \( \zeta_j \) component of normal

\[ F(x) \cdot \zeta \]

are divergence theorem, get laplacian

Will obtain the Darboux equation

Prove

\[ \left( \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_h(x, r) = \Delta_x M_h(x, r) \]