We want to solve a Dirichlet problem, finding $u$ given that $\Delta u = 0$ in $\Omega$, $u = f$ on $\partial \Omega$.

To do this, we need some tools.

A function is subharmonic if $u(x_0) \leq \frac{1}{V_{B_R(x_0)}} \int_{B_R(x_0)} u(x) \, dx$, where $V_{B_R(x_0)}$ is the volume of the ball centered at $x_0$. This is less than the average value on the sphere centered at $x_0$. This is also true of the average value in the ball.
Maximum Principle (strong version)

\[ \Omega \text{ open, bounded, connected, } u \in C(\Omega), \ u \text{ subharmonic in } \Omega \]

Then \( u \) is either
- constant in \( \Omega \)
- \( u(x) < \sup_{\Omega} u \) for all \( x \in \Omega \)

(If superharmonic, \( u(x) > \inf_{\Omega} u \))

Proof

If \( \sup_{\Omega} u = +\infty \) \( \forall x \), we are done.

Otherwise, \( \sup_{\Omega} u < +\infty \)

\[ \exists x_0 \in \Omega \text{ s.t. } u(x_0) = \sup_{\Omega} u \]

Let \( B_{R_0}(y) \subset \Omega \), \( y \in E \)

Claim: \( E \) is open

\[ u(y) \leq \int_{B_{R_0}(y)} u(x) \, dx \leq u(x_0) = u(y) \]

so \( u(y) = \int_{B_{R_0}(y)} u(x) \, dx \Rightarrow u(x) \equiv u(y) \) for \( x \in B_{R_0}(y) \)

thus \( B_{R_0}(y) \subset E \)

Now \( \Omega = E \cup \{ y \in \Omega : u(y) < u(x_0) \} \)

open \( \Omega \) open and \( u \) continuous

so \( \Omega \) connected, union of disjoint open sets thus

\[ E = \emptyset \text{ or } E = \Omega \]

Subharmonic functions, like harmonic ones, have a maximum value principle.

A subharmonic function is either constant, or is less than the maximum value, which is achieved on the boundary.

For superharmonic functions, greater than the minimum, on the boundary.
Some Properties of Subharmonic Functions

\[ \Sigma(\Omega) = \{ v \in C(\Omega) : v \text{ is subharmonic in } \Omega \} \]
\[ \sigma(\Omega) = \{ v \in C(\Omega) : v \text{ is superharmonic in } \Omega \} \]

**Proposition**

Let \( v, v_i \in \sigma(\Omega) \), \( i = 1, \ldots, N \) then

1. \( v \in \sigma(\Omega') \) for all subdomain of \( \Omega \)
2. \( \sum_{i=1}^{N} c_i v_i \in \sigma(\Omega) \) for \( c_i \geq 0 \), \( i = 1, \ldots, N \)
3. \( \max\{ v_1(x), \ldots, v_N(x) \} \in \sigma(\Omega) \)
   - This is the max of \( v_i \) at each point (pointwise max)
4. if \( f: \mathbb{R} \to \mathbb{R} \) nondecreasing, convex
   \[ \Rightarrow f(v) \in \sigma(\Omega) \]

**Proof**

1. trivial
2. follows from linearity of integral
3. \( x \in \Omega \) \( M(x) = \max\{ v_1(x), \ldots, v_N(x) \} \)
   \[ m(y) = v_j(y) \leq f(v_j(x)) dx = f(M(x)) d\sigma(x) \]

A subharmonic function is subharmonic on any subdomain.
Linear combinations of subharmonic functions are subharmonic.
The function \( m(x) = \max\{ v_1(x), \ldots, v_N(x) \} \), the pointwise maximum value of a finite family of subharmonic functions, is subharmonic.

If \( f: \mathbb{R} \to \mathbb{R} \) convex, \( f(v(x)) \) is subharmonic.
Jensen's Inequality

\[ f: \mathbb{R} \rightarrow \mathbb{R} \text{ nondecreasing, convex} \]

\[ E \subset \mathbb{R}^n \text{ measurable set, } |E| < \infty \]

\[ u \in L^1(E) \]

\[
\int_E \left( \frac{1}{|E|} \int_E u(x) \, dx \right) \, dx \leq \frac{1}{|E|} \int_E f(u(x)) \, dx
\]

follows from convexity

for all \( f \) through \( x \) where \( \frac{d}{dx} \) is a supporting hyperplane

i.e. line through \( x \) s.t. \( f(x) \geq \text{line}(x) \)

\[ \text{Note: this is true for any measure.} \]

so for \( (d) \)

\[
\frac{\partial}{\partial y} f(u(x) \delta_{x,y}) \leq f(u(x)) \delta_{x,y}
\]

\[
f(u(x)) \leq f(\int \frac{\partial}{\partial y} u(x) \delta_{x,y} \, dx) \leq f(\int \frac{\partial}{\partial y} u(x) \, dx) \]

\[
\text{nondecreasing} \quad \text{Jensen's}
\]
Dirichlet's problem for Laplace's equation in a ball solved using Poisson's formula

**Harmonic Lifting**

Let $B$ be a ball in $\mathbb{R}^n$ with $f \in C(\partial B)$.

**Existence of Potential**

$$\exists u \in C(\overline{B}) \cap C^2(B)$$

s.t.

- $\Delta u = 0$ in $B$
- $u = f$ on $\partial B$

**Solving for $u(x)$**

$$u(y) = \int_{\partial B} P(y, x) f(x) d\sigma(x)$$

$P$ is the Poisson kernel

**Green's Function for the Ball**

Let $v \in C(\Omega)$, $B$ a ball with $B \subset \Omega$.

**Existence of Potential**

$\exists v \in C(\overline{B})$ harmonic in $B$, continuous in $\overline{B}$

and $H_{v, B}(x) = v(x) \quad \forall x \in \partial B$

**Harmonic Lifting**

The harmonic lifting of $v$ in the ball $B$ is defined as

$$v_B(x) = \begin{cases} v(x) & x \in \Omega \setminus B \\ H_{v, B}(x) & x \in B \end{cases}$$

If solving for $u$ with $B$ a ball, $\Delta u = 0$ in $B$, and $u = f$ on $\partial B$,

then $u(x) = \int_{\partial B} P(x, y) f(y) d\sigma(y)$, where $P(x, y)$ is Poisson's kernel.

The harmonic lifting of a function $v \in C(\Omega)$ w.r.t. $B_\delta(x) \subset \Omega$ is the function harmonic in $B$, equal to $v$ in $\Omega \setminus B$ (including $\partial B$).
lifting of a smaller function to smaller than lift of larger function

The lift is nondecreasing

repeatable

subharmonic

\begin{align*}
\text{Let } u \leq v \\
\forall x \in \Omega \setminus B, \quad u_B(x) = u(x) \quad \text{if } x \in \Omega \setminus B \\ 
v_B(x) = v(x) \quad \text{if } x \in \Omega \setminus B \quad \Rightarrow \quad u_B(x) \leq v_B(x), \quad \forall x \in B \\

x \in B \\
u_B(x) \text{ harmonic in } B \\
v_B(x) \text{ harmonic in } B \\

u_B(x) - v_B(x) \text{ harmonic in } B \\

x \in \partial B, \quad \frac{\partial}{\partial n}(u(x) - v(x)) = u(x) - v(x) \leq 0 \\

\Rightarrow \quad \frac{\partial}{\partial n} \left( u_B(x) - v_B(x) \right) = \frac{\partial}{\partial n} \left( u(x) - v(x) \right) \leq 0 \\

\text{max}_{\partial B} \left( u_B(x) - v_B(x) \right) = \text{max}_{\partial B} \left( u(x) - v(x) \right) \leq 0 \\

\text{maximum principle} \\

\text{Thus, } u_B(x) \leq v_B(x) \quad \forall x \in B \\

\forall x \in \Omega \\

\text{Proposition} \\
\text{Suppose } v \in \mathcal{C}^1(\Omega) \quad \text{and } \quad \overline{B} = \overline{B(\bar{z})} \subset \Omega \\
\text{Then} \\
a) \quad v(x) \leq v_B(x) \quad \forall x \in \Omega \\
b) \quad B' \text{ ball, } \quad \overline{B'} \subset \Omega \quad \Rightarrow \quad (v_B)_{B'}(x) \geq v_B(x), \quad \forall x \in \Omega \\
c) \quad v_B(x) \in \mathcal{C}(\Omega) \\

\text{Proof} \\
a) \quad v \in \mathcal{C}(B) \quad v_B \text{ harmonic in } B \quad \Rightarrow \quad u - v_B \in \mathcal{C}(B) \\
\text{because } \overline{B'} \subset B \quad \text{so } u \in \mathcal{C}(B), \quad \text{so less than average, } v_B = \frac{\int_B v_B}{B} \\
\text{so } u(x) - v_B(x') \leq \int_{B'} u(x) dx - \int_{B'} v_B dx = \int_{B'} u - v_B dx \quad \checkmark \\

\text{We have } \quad v_B(x) \geq v(x) \\
\quad \left( v_B \right)_{B'}(x) \geq v_B(x) \\
\text{and } \quad v_B(x) \text{ is subharmonic}
From here, apply maximum principle to see:

\[ U - V_0 \leq \max_{\partial \Omega} (U - V_0) \quad \text{(as } U - V_0 \text{ is constant)} \]

\[ = 0 \quad \text{since } V(x) = V_0 \text{ on } \partial \Omega \text{ since } V_0 \text{ is a harmonic function.} \]

Thus \[ V - V_0 \leq 0 \quad \Rightarrow \quad V \leq V_0 \text{ in } B_{\delta/2} \]

b) is straightforward by induction.

c) for all \( x_0 \in \Omega \), \( B_{\delta/2}(x_0) \subset \Omega \) to show

\[ V_0(x_0) \leq \int_B v(x) \, d\sigma(x) \]

2 possibilities

1. \( x_0 \in \Omega \setminus B \)

2. \( x_0 \in B \)

1. \( x \in \Omega \setminus B \) (a) \[ v \text{ subharmonic, } \quad \text{since } V = V_0 \text{ in } \Omega \setminus B. \]

\[ \int_{B_{\delta/2}(x_0)} v(x) \, d\sigma(x) \geq \int_{B_{R}(x_0)} v(x) \, d\sigma(x) \geq V(x_0) = V_0(x_0) \]

so if \( x_0 \) not in the ball \( B \), done.
2. Suppose \( x_0 \in B = B_{r}(\overline{y}) \)
   by contradiction, \( v_{B}(x_0) > \int_{\partial B_{r}(x_0)} v_{B}(x) \, d\sigma(x) \)

Let again, let
\[
W(x) = \begin{cases} 
  v_{B}(x) & x \in \Omega \setminus B_{r}(x_0) \\
  H(x) & x \in B_{r}(x_0) 
\end{cases}
\]

By (a) \( W(x) \geq v_{B}(x) \) for all \( x \in \Omega \)
and \( W(x) \) is harmonic \( x \in B_{r}(x_0) \)

\[
W(x_0) = \int_{\partial B_{r}(x_0)} W(x) \, d\sigma(x) = \int_{\partial B_{r}(x_0)} H(x) \, d\sigma(x)
\]

Thus, \( \int_{\partial B_{r}(x_0)} H(x) \, d\sigma(x) \leq \int_{\partial B_{r}(x_0)} v_{B}(x) \, d\sigma(x) \)

Thus \( v_{B}(x_0) - W(x_0) > 0 \)

will get contradiction next time.