Today: related to Stoke's theorem

\[ F(x) = (F_1(x), ..., F_n(x)) \quad x \in \mathbb{R}^n \quad \text{smooth, etc} \]

particle \( \sigma(t) \) is moving under the action of \( F \).

\( \dot{\sigma}(t) \) is position of particle at time \( t \)

Newton's law: sum of all forces acting = mass \( \times \) acceleration

\[ F(\sigma(t)) = m \ddot{\sigma}(t) \]

\[ \dot{\sigma} = (\dot{\sigma}_1, ..., \dot{\sigma}_n) \]

\[ \ddot{\sigma} = (\ddot{\sigma}_1, ..., \ddot{\sigma}_n) \]

\[ \dddot{\sigma} = (\dddot{\sigma}_1, ..., \dddot{\sigma}_n) \]

dot product with \( \dot{\sigma}(t) \)

\[ F(\sigma(t)) \cdot \dot{\sigma}(t) = m \ddot{\sigma}(t) \cdot \dot{\sigma}(t) = \frac{m}{2} \frac{d}{dt} \left( \dot{\sigma}^2(t) \right) \]

\[ T(t) = \frac{1}{2} m \dot{\sigma}^2(t) \quad \text{Kinetic energy of the particle at time } t \]

\[ F(\sigma(t)) \cdot \dot{\sigma}(t) = \frac{d}{dt} (T(t)) \]

\[ \int_{t_0}^{t_1} F(\sigma(t)) \cdot \dot{\sigma}(t) \, dt = T(t_1) - T(t_0) \]

\[ \text{work of the particle from } \sigma(t_0) \text{ to } \sigma(t_1) \]

\[ C \text{ is the path of } \sigma \]

Newton's law \( F(\sigma(t)) = m \ddot{\sigma}(t) \)

Kinetic energy \( \frac{1}{2} m \dot{\sigma}^2(t) = T(t) \)

The work on a particle \( \int_{t_0}^{t_1} F(\sigma(t)) \cdot \dot{\sigma}(t) \, dt = \frac{d}{dt} T(t) \)
Def \[ F \text{ is conservative if} \]
\[ \int_{t_0}^{t} F(\sigma(t)) \cdot \sigma'(t) \, dt \]
is independent of the path \[ \sigma(t) \].

F is conservative \[ \iff \exists \phi(t) \text{ scalar} \mid F = \nabla \phi \]
\[ \phi \text{ is called a potential} \]

Proof
\[ \iff \text{Suppose } \sigma(t) \text{ is a path from } \sigma(t_0) = p_0 \text{ to } p_1 = \sigma(t_1) \]
\[ f(t) = \phi(\sigma(t)) = D\phi(\sigma(t)) \cdot \sigma'(t) \]
\[ \Rightarrow \int_{t_0}^{t} F(\sigma(t)) \cdot \sigma'(t) \, dt = \int_{t_0}^{t} f(t) \, dt = \phi(t_1) - \phi(t_0) \]
\[ = \phi(p_1) - \phi(p_0) \]

\[ \Rightarrow \text{Construct the potential} \]
\[ p_0, p_1 \text{ points} \]
\[ W(p_0, p_1) = \int_{p_0}^{p_1} F \] is independent of \[ \sigma(t) \].

\[ \text{Path:} \quad W(Q, p_0) = W(Q, p) + W(p, p_0) \]
\[ H = (h, 0, \ldots, 0) \]
\[ W(p+h, p) = W(p+h, p_0) - W(p_0) \]
\[ \sigma(t) = p + tH \quad 0 \leq t \leq 1 \]
\[ \sigma(t) = p + tH \quad W(p+h, p) = \int_{0}^{1} F(p + tH) \, dt = [\sigma(1)]_0^1 \]

F is conservative if \[ \int_{t_0}^{t} F(\sigma(t)) \cdot \sigma'(t) \, dt \] does not depend on \[ \sigma(t) \].

F is conservative iff \[ \exists \phi : \mathbb{R}^n \to \mathbb{R} \mid F = \nabla \phi \] call \[ \phi \] the potential.
\[ \frac{W(P + h, P_0) - W(P, P_0)}{h} = \int_0^1 F'(P + th) \, dt \]

\[ \lim_{h \to 0} \longleftarrow F'(P) \]

\[ \frac{\partial}{\partial x_i} W(P) = F_i(P) \]

Given componentwise, we get
\[ \phi(P) = W(P) \]
\[ DW = F \]

\[ F(x) = \frac{x - P}{|x - P|^3} \] is conservative.

\[ \phi(x) = \frac{1}{|x - p|} \] is potential (specific, the Newtonian potential).

\[ \nabla \phi = C \cdot F(x) \]

\[ \text{Curl } F(x, y, z) \text{ is a field in } \mathbb{R}^3 \]

\[ C \text{ is a closed path in } \mathbb{R}^3 \]

Work done by F along C is \( \int_C F \cdot ds = \int_C (\phi(0) - \phi(1)) \, dt \)

is called the circulation of F along C.

The circulation is \( \int_C F \cdot ds \).

The curl is the limit of the circulation normalized by the area as C approaches a point in a specified surface.
\[ \lim_{C \to P} \frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{s} = \nabla \times \mathbf{F} \]

What is the meaning of the curl?

Suppose \( \mathbf{F} \) is parallel to \( y \) axis.

\( C \) is a circle in plane parallel to \((XZ)\)

\[ \mathbf{F} = (F_1, F_2, F_3) = (0, F_z, 0) \]

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_z(\sigma(t)) \cdot \hat{n}(t) \, dt = 0 \]

\( \sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_3(t)) \)

\( \dot{\sigma}(t) = (\dot{\sigma}_1(t), \dot{\sigma}_2(t), \dot{\sigma}_3(t)) \)

Take \( C \) parallel to \((yz)\) plane

\( \sigma(t) = (A, \sigma_2(t), \sigma_3(t)) \)

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_C F_z(A, \sigma_2(t), \sigma_3(t)) \cdot \sigma_2(t) \, dt \]

like a paddlewheel in a fluid.

\[ \text{Curl is "amount of rotation in a plane at a point."} \]
F any field (surf, smooth)
\[ F = \nabla \phi + (\nabla \times G) \]

\[ \nabla \times F = \begin{vmatrix}
  i & j & k \\
  \partial_x & \partial_y & \partial_z \\
  F_1 & F_2 & F_3 
\end{vmatrix} \]

curl

Helmholtz decomposition

\[ \text{curl}_n \mathbf{F}(p) = \mathbf{n} \times (\nabla \times \mathbf{F})(p) \]

Note

\[ \text{Stokes} \quad \text{Thm} \]

\[ \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{S} \mathbf{n}(p) \times (\nabla \times \mathbf{F})(p) \, d\sigma(p) \]

The Helmholtz decomposition of a field into \( \mathbf{F} + (\nabla \times \mathbf{G}) \) where \( \mathbf{F} \) conservative.

\[ \nabla \times \mathbf{F} = \begin{vmatrix}
  i & j & k \\
  \partial_x & \partial_y & \partial_z \\
  F_1 & F_2 & F_3 
\end{vmatrix} \]

\[ \text{curl}_n \mathbf{F}(p) = \mathbf{n} \times (\nabla \times \mathbf{F})(p) \]

Stokes theorem

\[ \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} \text{curl}_n \mathbf{F}(p) \, d\mathbf{s} = \int_{S} \mathbf{n}(p) \times (\nabla \times \mathbf{F})(p) \, d\sigma(p) \]
Thm \ F \text{ is conservative } \iff \nabla \times \mathbf{F} = 0

Proof \Rightarrow \exists \text{ potential } \phi \Rightarrow F = \nabla \phi

\text{but} \quad \nabla \times (\nabla \phi) = 0

\Leftarrow \text{ By Stokes}\]

\int_{C} \mathbf{F} \cdot d\mathbf{s} = 0

\text{But}\]

\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds = 0

\text{DIVERGENCE THEOREM}\]

F \in C^1(\mathbb{R}^3) \Rightarrow \int_{\Omega} \text{div} \ F(x) \, dx = \oint_{\partial \Omega} \mathbf{F}(x) \cdot n(x) \, d\sigma(x)

\Omega \text{ is a bounded domain (open smooth boundary)}

\text{GREEN'S FORMULAS}\]

\forall u, v \in C^2(\overline{\Omega}) \Rightarrow \text{\nabla^2} u \text{ Laplacian of } u

\int_{\Omega} (v \Delta u + \nabla u \cdot \nabla v) \, dx = \int_{\partial \Omega} v(x) \frac{\partial u(x)}{\partial n} \, d\sigma(x)

\text{\nabla^2} u \text{ is the normal derivative of } u

\Delta u(x) = n(x) \cdot \mathbf{n} \text{ outer and normal}
Proof

Let \( F(x) = v(x) Du(x) \) is a \( C^1 \) field in \( \Omega \)

\[
\int_\Omega \nabla \cdot F(x) \, dx = \int_\Omega v(x) Du(x) \cdot \eta(x) \, d\sigma(x)
\]

\[
= \int_\Omega v(x) \frac{\partial u(x)}{\partial \eta} \, d\sigma(x)
\]

\[
F_j = v(x) \frac{\partial u}{\partial x_j}(x)
\]

\[
\frac{\partial F_j}{\partial x_j}(x) = \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_j} + v(x) \frac{\partial^2 u}{\partial x_j^2}
\]

\[
\nabla \cdot F = \sum_{j=1}^n
\]

\[
= \nabla \cdot Du + v(x) \Delta u(x)
\]

2nd Green's Formula

\( u, v \in C^2(\bar{\Omega}) \)

\[
\int_\Omega (v \Delta u - u \Delta v) \, dx = \int_\Omega \left( v \frac{\partial u}{\partial \eta} - u \frac{\partial v}{\partial \eta} \right) \, d\sigma(x)
\]

\( \nabla \cdot \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \)?

Is \( u \) a radial solution?

Propose \( u(x) = \phi(r) \) where \( r = |x| \)

\[
\frac{\partial u}{\partial x_j}(x) = \phi'(r) \frac{dx_j}{dx}
\]

\[
\frac{\partial}{\partial x_j} \frac{1}{\sqrt{x^2}} = \frac{x_j}{x^2}
\]

\[
\int_\Omega v u u - u v u \, dx = \int_\Omega v \frac{\partial u}{\partial \eta} - u \frac{\partial v}{\partial \eta} \, d\sigma(x)
\]
\[ \frac{\partial^2 r}{\partial x_j^2} = \sqrt{x^2} - \frac{1}{\alpha} x_j (x^2)^{-\frac{3}{2}} \frac{\partial}{\partial x_j} \]

\[ \Delta u = \sum_{j=1}^{n} \frac{\partial^2 r}{\partial x_j^2} = n \frac{1}{x^1} - \frac{1}{x^1} - 3 \sum (x_j)^2 \]

\[ = \frac{n}{x^1} - \frac{|x|^2}{x^3} \]

\[ \frac{\partial^2 u}{\partial x_j^2} = \phi''(r) \]

finish next time

Next week no class

First midterm week after - Thursday, 21 Oct

Wednesday of that week: Wednesday 26 Oct

2 pm class (20 Oct 2010)

1st Exam Tuesday 26 October 2010

closed book