Thm \[a(x,y,z), b(x,y,z), c(x,y,z) \in C'(\Omega \times \mathbb{R})\]

\[x_0 \in \Omega, (x_0, y_0, z_0) \in \Omega \text{ point}\]

\[(f(s), g(s), h(s)) \in C'(15 - |s_o| < 8) \text{ continuous curve}\]

\[f(s_o) = x_0, g(s_o) = y_0, h(s_o) = z_0\]

Then there exists a unique solution \(z = u(x, y)\) to the equation

\[a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)\]

in a neighborhood \(N\) of \((x_0, y_0, z_0)\) \(\exists h(s) = u(f(s), g(s)) \text{ if } \det(J) \neq 0\)

\[J = \begin{vmatrix} a(x_0, y_0, z_0) & f'(s_0) \\ b(x_0, y_0, z_0) & g'(s_0) \end{vmatrix} \neq 0 \quad \text{if } \det(J) \neq 0\]

What happens when \(|J| = 0\)

will show \(J = 0 \Rightarrow (f(s), g(s), h(s))\) is parallel

\[\alpha \begin{pmatrix} a(x, y, z_0) \\ b(x, y, z_0) \end{pmatrix}, \underbrace{c(x, y, z_0)}_{\text{given solution exists}}\]

\[h(s) = u(f(s), g(s)), |s - s_0| < \delta\]

\[
\begin{cases}
  x' = h' = f'(s)x + f(s)u_x + g(s)u_y = 0 \\
  y' = a(x, y, z_0)u_x + b(x, y, z_0)u_y = c(x, y, z_0)
\end{cases}
\]

\[
\begin{vmatrix} a & f' \\ b & g' \end{vmatrix} \neq 0 \quad \text{for solution to exist/be unique}
\]

If \(J = 0\) then \((f(s), g(s), h(s))\) is parallel to \((a, b, c)\) i.e. vanishing field

\[a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)\]
So, if \( \Gamma \) is a curve for which \( T \) and \( \mathbf{c}(b) \) are linearly independent, then there is no solution.

So, \( T \) is a curve of solution space is I.

\[
A = \begin{bmatrix} b & -a \\ 0 & c \end{bmatrix}, \quad T = \begin{bmatrix} b' \\ 0 \end{bmatrix}, \quad A\mathbf{T} = 0
\]

\[
k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f(x, y) = 0, \quad g(x, y) = 0
\]

\[
f(y, x) = 0
\]
Uniqueness when $J=0$

$\Gamma$ is characteristic then there are infinitely many solutions.

\[
\begin{align*}
  f'(s_0) &= a(x_0, y_0, z_0) \\
  g'(s_0) &= b(x_0, y_0, z_0) \\
  h'(s_0) &= c(x_0, y_0, z_0)
\end{align*}
\]

Take \( \overline{\Gamma} = (f(s), g(s), h(s)) \)

\[
\begin{align*}
  f(s) &= a(s - s_0) + x_0 \\
  g(s) &= b(s - s_0) + y_0 \\
  h(s) &= \text{any } C^1 \text{ function with } h(s_0) = z_0
\end{align*}
\]

Solve \((x)\) \( h(s) = u(f(s), g(s)) \)

\[
\begin{align*}
  f'(s_0) &= a \\
  g'(s_0) &= b \\
  \left| \begin{array}{cc} a(x_0, y_0, z_0) & a \\
  b(x_0, y_0, z_0) & b \end{array} \right| &= a\beta - a\beta \neq 0 \\
  &\quad \Rightarrow (a, b) \cdot (\beta - \alpha) \neq 0
\end{align*}
\]

Thus, there are infinitely many $\alpha, \beta \in \mathbb{R}$, $J \neq 0$.

Let $u_{\alpha \beta}$ be a solution of $(x)$ passing through $\overline{\Gamma}$.

Claim: $u_{\alpha \beta}$ contains $\Gamma$.

\[
\begin{align*}
  u_{\alpha \beta}(x_0, y_0) \cdot u_{\alpha \beta}(f(s), g(s)) &= h(s) \\
  u_{\alpha \beta}(x_0, y_0) &= z_0
\end{align*}
\]
Find $z = u(x,y)$ that solves

$$F(x, y, u(x,y), u_x(x,y), u_y(x,y)) = 0$$

for $(x,y)$ in a neighborhood of $(x_0, y_0)$

What this means

- $(x, y, z)$ point
  
  $p = u_x(x,y)$
  $\ell = u_y(x,y)$

$$
\begin{align*}
\xi - u(x,y) &= p(\xi - x) + \ell(\eta - y) \\
\ell &= u(x,y)\text{ at } (x,y)
\end{align*}
$$

$x$th $F(x, y, u(x,y), p, \ell) = 0 =: G(p, \ell)$

- the tangent plane satisfies (**)  
  
- fix $(x, y, u(x,y))$ look at all $p, \ell$ st. (**) 
  
- it is a curve in $p, \ell$ in subspace, is line 

- the envelope of all tangent planes to $u$ at $(x,y, u(x,y))$ are in a "cone,

- this will be tangent to the cone at each point
\((u_x)^2 + (u_y)^2 = 3\) so \(p^2 + r^2 = 3\) \(\Rightarrow\) circle of radius 3

So \((p, r)\) photon curve describes cross section of cone

e.g. \(p^2 + 2r^2 = 3\) elliptical

---

How to Solve this Analytically

Suppose \(F(x, u(x), Du(x)) = 0\) \((1)\)

\[X \in \mathbb{R}^n\]

\[u(x) \in \mathbb{R}^n\]

\(Du(x) = (u_{x_1}(x), \ldots, u_{x_n}(x)) \in \mathbb{R}^n\)

And \(z = u(x)\) is a solution to \(F(x, u(x), Du(x)) = 0, x \in \Omega\)

write \(p(x) = Du(x)\) so \(p_i(x) = u_{x_i}(x)\) \(i = 1, \ldots, n\)

Differentiate \((1)\) w.r.t. \(x_j\)

will get another \(\alpha_1, \ldots, \alpha_n\) in a system

\[0 = F_{x_j}(x, u(x), Du(x)) + F_u(x, u(x), Du(x)) u_{x_j}(x)\]

\[+ \sum_{k=1}^n F(x, u(x), Du(x)) \cdot u_{x_k} \cdot x_k(x)\]

\(\alpha_{k+1}\) derivative of \(u_{x_k}\) and \(x_k\)

Take any curve \(x = x(t)\)  so

set \(z(t) = u(x(t))\)  \(p_j(t) = u_{x_j}(x(t))\)  \(j = 1, \ldots, n\)

\[\frac{dz}{dt} = \dot{z} = \sum_{k=1}^n u_{x_k}(x(t)) \cdot x_k'(t) = \sum_{k=1}^n p_k(t) \cdot x_k'(t)\]
\[ p_j'(t) = \sum_{k=1}^{n} U_{x_j x_k} \left( x(t), z(t) \right) \cdot X_k'(t) \]

Choose \( x(t) \) such that

\[ X_k'(t) = F_{x_k}(x, z(t), p(t)) \]

Insert into equation

\[ 0 = F_{x_j}(x(t), z(t), p(t)) + F_u(x(t), z(t), p(t)) p_j(t) \]

\[ + \sum_{k=1}^{n} U_{x_j x_k} \left( x(t), z(t) \right) X_k'(t) \]

Sum is just \( p_j'(t) \) so

\[ p_j'(t) = -F_{x_j}(x(t), z(t), p(t)) - F_u(x(t), z(t), p(t)) p_j(t) \]

\[ j = 1, \ldots, n \]

\[ p'(t) = -F_x(x(t), z(t), p(t)) - F_u(x(t), z(t), p(t)) p(t) \]

Vectors of vectors - scalar

\[ X' = p(t) \cdot x'(t) \]

\[ = p(t) \cdot F_p(x(t), z(t), p(t)) \]

Scalar - 1 term

\[ \dot{x} = F_p(x(t), z(t), p(t)) \]

Vector - \( n \) terms

So we have \( 2n + 1 \) ode's together the characteristic equations

Homework postponed to Thursday
- Bookmarks
- Homework pages
- Bring paper
- Green pen?
- 3 hds & adaptors
  - fast, med/slow

\[ \alpha x + \beta y = c \]
\[ z = u(x, y) \]

U
- earhds
- iPod docked
- Kindle paper, etc.
PDE

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