Algebra 2011-04-14 p1

[Email our schedules to hosteller.]

\[ \mathbb{R}[x,y] \] deformation ring, gen by \( xy \)
deform \( xy - yx = 1 \) instead of \( xy - yx = 0 \)

**Thm** \( \text{of 460-462} \)
Assume \( R \) a PID, \( M \) is free of rank \( n < \infty \) let \( N \) be a zero submodule of \( M \). Then

1. \( N \) is free of rank \( m \) \( \geq n \)
2. There exists a basis \( y, \ldots, y_n \) for \( M \) s.t.
   \[ a_1, a_2, \ldots, a_m \] is a basis for \( N \) where
   \[ a_1, \ldots, a_m \] nonzero elements of \( R \) s.t.
   \[ a_1 | a_2 | \ldots | a_m \]

**Proof** for each \( R \) module homomorphism \( M \rightarrow R \)
choose \( a_0 \in R \) s.t. \( \langle N \rangle = (a_0) \) an ideal of \( R \) (see last time)

Set \[ \Sigma = \{ (a_0) | \phi \in \text{Hom}_R(M,R) \} \]
Saw last time that \( \Sigma \) is a nonempty and contains an ideal maximal among ideals in \( \Sigma \). So
homomorphism \( \varphi : M \rightarrow R \) s.t. \( \varphi(N) = (a_0) \) is a
maximal member of \( \Sigma \).

Set \( a_1 = a_0 \in R \), choose \( y \in N \) s.t. \( \varphi(y) = a_1 \),
\[ a_1 \neq 0 \]
$y_i$ can be taken as part of basis of $M$
$a_i y_i$ as part of basis of $N$

top of p 462
By (a) $R a_i y_i \cap (N \cap \ker z) = 0$
and so
$N = Ra_i y_i \oplus \ker z$

Part (b) follows

Since $N \neq 0$, rank $N = m > 1$
Since $N = Ra_i y_i \oplus \ker z$

it follows $\ker z \leq m - 1$

by induction, assume $\ker z$ is free (possibly $\{0\}$)

$Ra_i y_i \cong R$ as left $R$-module (exercise)

so $Ra_i y_i \oplus \ker z$ since direct sum of free $R$-modules is free (exercise)

Exercise $M = Ra_i y_i \oplus \ker z$

Where the hell did this come from?

Examples Let $M$ be a free $\mathbb{Z}$ module of rank $n$.
let $N$ be a submodule, also of rank $n$.
By this theorem, there is a basis $y_1, \ldots, y_n$ for $M$ and non-negative integers $a_1, \ldots, a_n$ such $y_1, \ldots, y_n$ as basis for $M$, $a_1 y_1, \ldots, a_n y_n$ is basis for $N$.

identify $M$ with $\mathbb{Z}^n$, then $N = X \oplus Y \oplus \ldots \times \mathbb{Z}$, etc.

NB. $M / N \cong (\mathbb{Z} / a_1 \mathbb{Z}) \times \ldots \times (\mathbb{Z} / a_n \mathbb{Z})$
Continue we let \( R \) be a PID, and let \( C \) be a cyclic \( R \)-module. So \( \pi \) is a \( C \in C \) st. \( \pi \) is a left \( R \)-module map

\[
\begin{align*}
R & \longrightarrow R \cdot c = C \\
\pi & \longrightarrow \pi \\
\end{align*}
\]

is surjective into \( C \) i.e. \( R \cdot c = C \). Now

\[
\frac{R}{\ker \pi} \cong C.
\]

Since \( R \) is a PID, we may write \( \ker \pi = (a) \) for some \( a \in R \).

\[
\frac{R}{(a)} \cong C \quad \text{as a left } \ R \text{-module}
\]

Also \( \ker \pi = \text{ann}_R C = \text{ann}_R C \), see last line.

**FUNDAMENTAL THEOREM OF MODULES OVER PIDS**

Existence Part.

Let \( R \) be a PID and \( M \) a finitely generated \( R \)-module.

Then

1. \( M \cong R \oplus \frac{R}{(a_1)} \oplus \ldots \oplus \frac{R}{(a_m)} \)

   for some integer \( r \geq 0 \) and nonzero \( a_1, \ldots, a_m \in R \) not units st.

   \( a_1 | a_2 | \ldots | a_m \)

2. \( M \) is torsion free \( \iff \) \( M \) is free

3. In the decomposition 1

   \[
   \text{Tor}_1(M) \cong \frac{R}{(a_1)} \oplus \frac{R}{(a_2)} \oplus \ldots \oplus \frac{R}{(a_m)}
   \]

   as \( R \)-modules, and \( \text{ann}_R \text{Tor}_1(M) = (a_m) \).