Aside

Let \( R \) be ring 1/1

A ring with homomorphism from \( R \) to center of \( A \)

We say \( A \) with this homomorphism is an \( R \)-ALGEBRA

This gives \( A \) structure as \( R \)-\( R \)-bimodule, \( RA \)-bimodule, \( AR \)-bimodule via

\[
ra = \varphi(r)a \quad \text{where } \varphi \text{ is the homomorphism } R \to A
\]

Now suppose \( A, B \) are both \( R \)-algebras. Then we can form \( A \otimes_R B \). Saw in class that \( A \otimes_R B \) is a left \( A \)-module. Similarly, is a right \( B \)-module.

Prop: \( A \otimes_R B \) is a ring via

\[
(a \otimes b)(a' \otimes b') = aa' \otimes bb' \quad \text{for } a, a' \in A \quad b, b' \in B.
\]

Of see text on 

always think of fundamental theorem of finite abelian groups

\[ \mathbb{G} \otimes \mathbb{Q} : \mathbb{Z} : [i] \otimes \mathbb{Z} R \cong \mathbb{C} \]

§12.1

\( R \) commutative integral domain. Also \( M \) is a left \( R \)-module

Prop: Suppose \( M \) is free of rank \( n < \infty \), let \( y_1, \ldots, y_n \in M \).

Then \( y_1, \ldots, r_{n+1} \in R \) st, \( x_i \in R \) st.

\[ r_{n+1} y_{n+1} = 0 \]
Next, Recall \( \text{Tor}(M) = \{ x \in M \mid r x = 0 \text{ for some } r \neq 0, r \in R \} \)

1) The Torsion Submodule of \( M \)

N.B. A torsion submodule is a submodule of \( \text{Tor}(M) \)

2) If \( \text{Tor}(M) = 0 \) then \( M \) is torsion-free

3) \( N \) submodule of \( M \),

\[
\text{Ann}_R(N) = \{ r \in R \mid r.N = 0 \}
\]

If \( N \) is not torsion, then \( \text{Ann}_R N = 0 \)

that is, if \( \text{Ann}_R N \neq 0 \) then \( N \) is torsion

Saw in homework that \( \text{Ann}_R N \) is an ideal of \( R \).

4) Suppose \( R \) is a PID and \( N \subseteq M \) as submodules. Then

\[
\text{ann}_R L = \text{ann}_R N, \quad \text{and} \quad \text{ann}_R L = (b) \quad \text{ann}_R N = (b)
\]

then \( a | b \) since \( (b) \subseteq (a) \)

5) \( R \) still PID, \( m \in M \) Note

\[
\text{Ann}_R m = \{ r \in R \mid r.m = 0 \}
\]

\[
= \text{Ann} \left( R.m \right) = (a)
\]

by above if \( (b) = \text{Ann}_R M \), then \( \text{Ann}_R a | b \)

6) Now suppose \( R = \mathbb{Z} \) also suppose \( |M| = n < \infty \)

Let \( (b) = \text{Ann}_R M \), for some \( b \in \mathbb{Z} \)

Since \( n \in \text{Ann}_R M \), \( b | n \)

Choose \( x \in M \) and set \( (a) = \text{Ann}_R (R.x) = \text{Ann}_R x \), Then \( (b) \subseteq (a) \)

and so \( a | b \). \( : a | n \).

That is the order of \( x \) as an element of the abelian group \( M \) (i.e., \( \mathbb{Z} \)-module \( M \)) divides \( n \), the order of \( M \). This is Lagrange's Theorem.
**Def.** The **rank** of $M$ as a left module over arbitrary integral domain $R$ is the maximum number of $R$-linearly independent elements of $M$.

**Theorem.** Assume $R$ is a PID, and suppose $M$ is free of rank $n < \infty$. Let $N$ be a nonzero submodule of $M$.

Then:

1) $N$ is free of rank $m \leq n$

2) There is a basis $y_1, y_2, \ldots, y_m$ of $M$ s.t.

$$a_1y_1, a_2y_2, \ldots, a_my_m$$

is a basis of $N$ where $a_1, a_2, \ldots, a_m$ nonzero elements of $R$ with divisibility relations $a_1 | a_2 | \ldots | a_m$.

**Proof.** To start, let $X$ be a nonempty collection of ideals of the PID $R$.

Saw last semester that every ascending chain of ideals in $X$ stabilizes, that is, for some positive $N$,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$$

Eventually stabilizes, that is, for some positive $N$,

$$I_N = I_{N+1} = \ldots$$

that is, Noetherian.

It follows (cn) that $X$ must contain an ideal maximal among ideals in $X$.

For each $R$-module homomorphism $M \to R$, $\langle \hat{\phi}(N) \rangle$ is a submodule of $R$ that is an ideal of $R$. So $\langle \hat{\phi}(N) \rangle = \langle a \phi \rangle$ for some $(a \phi) \in R$.

Set $\Sigma = \{ (a \phi) \mid \phi \in \text{Hom}_R (M, R) \}$

Note $(0) \in \Sigma$ since one can choose $\phi$ to be the zero map, so $\Sigma$ is not empty.

So by above, must exist a maximal element of $\Sigma$. 

So there is a homomorphism
\[ \varphi : M \rightarrow R \]

such that \( \varphi(N) = (a_1) \) is not contained in any other ideal in \( \Sigma \).

Set \( a_i = a_{yi} \in R \) and choose \( y \in N \) s.t. \( \varphi_i(y) = a_i \).

We show \( a_i \neq 0 \). Take \( x_1, \ldots, x_n \)

be any basis for \( M \) and let
\[ \pi_i : M \rightarrow R \]

be the map sending
\[ r_1 x_1 + \ldots + r_n x_n \mapsto r_i \]

(Note: well defined since \( x_1, \ldots, x_n \) is a basis.)

Since \( N \neq \{0\} \), exists some \( i \) such that \( \pi_i(N) \neq 0 \)

Since \( \pi_i(N) \in \Sigma \) we see that \( \Sigma \not\subset \{0\} \)

\[ a_i \neq 0 \]

Next, let \( \Psi : M \rightarrow R \) be any \( R \)-module homomorphism, and let

\[ y \in N \). We show that \( a_i \mid \Psi(y) \) as follows:

consider the ideal \( (a_i, \Psi(y)) \) of \( R \). Since \( R \) is a PID, \( (a_i, \Psi(y)) = (d) \)

Also \( d = r_1 a_i + r_2 \Psi(y) \) for some \( r_1, r_2 \in R \). \[ d \mid a_i \text{ and } d \]

Take \( \Psi : m \mapsto r_1 \varphi(m) + r_2 \Phi(m) \) so \( \Psi(y) = r_1 a_i + r_2 \Psi(y) = d \).

\[ d \in \Psi(N) \]

\[ (d) \subseteq \Psi(N) \]

But \( (a_i) \subseteq (d) \subseteq \Psi(N) \) since \( (a_i) \) maximal, all equal, so \( a_i \mid \Psi(y) \)

In particular \( a_i \mid \pi_i(y) \) for all \( i \). Write \( \pi_i(y) = a_i b_i \) for

suitable \( b_i \), all \( i \).