Algebra 2011-04-04 p1

Brief Introduction to Algebraic Geometry

Throughout, let \( K \) be a field.

Most basic idea: Topology on \( K^n \)
determined by algebraic (i.e. polynomial) equations.

Textbooks usually do this with ring theory. Interesting to do bare-handed.
We'll think of polynomials as \( K[X_1, \ldots, X_n] \) both as elements in this ring and as functions \( K^n \to K \) given by evaluation
\[ (\lambda_1, \ldots, \lambda_n) \mapsto f(\lambda_1, \ldots, \lambda_n) \]
Now set \( \mathcal{O} = K[X_1, \ldots, X_n] \). For \( f_1, \ldots, f_n \in \mathcal{O} \) take
\[ V(f_1, \ldots, f_n) \text{ to be } \]
\[ \{ (\lambda_1, \ldots, \lambda_n) \in K^n \mid f_1(\lambda_1, \ldots, \lambda_n) = f_2(\lambda_1, \ldots, \lambda_n) = \ldots = f_n(\lambda_1, \ldots, \lambda_n) = 0 \} \]
Note that \( V(1) = \emptyset \) and \( V(0) = K^n \)
More generally, for \( S \subseteq \mathcal{O} \), have \( V(S) = \{ (\lambda_1, \ldots, \lambda_n) \in K^n \mid f(\lambda_1, \ldots, \lambda_n) = 0 \} \)

Prop: The sets \( V(S) \), for \( S \subseteq \mathcal{O} \), taken as the closed subsets, form a topology on \( K^n \).
If we've seen that \( \emptyset \) and \( K^n \) both closed.

Let \( \{ S_i \} \) be arbitrary subsets of \( \mathcal{O} \). Take \( S = \bigcup_i S_i \). Then
\[ \bigcap_i V(S_i) = V(\bigcup_i S_i) = V(S) \text{ is a closed set (countable intersection of closed)} \]
Finite unions remain closed (ex).
By easy induction, suffices to show that \( V(S) \cup V(T) \) closed for \( S, T \subseteq \mathcal{O} \).
For $V(S) \cup V(T)$ take $U = \{ fg \mid f \in S, g \in T \}$ then

$$V(S) \cup V(T) = V(U)$$

and so is closed

\[ U \subseteq V(U) \subseteq V(S) \cup V(T) \] by integral domain property of $\Omega$

**Def** We call the sets $V(S)$ for $S \subseteq \Omega$ as ALGEBRAIC SETS.

(Sometimes no varieties, though this also has a more specific meaning.)

Refers to the above topology as the ZARISKI TOPOLOGY on $K^n$.

**Prop** let $S \subseteq \Omega$. Then $V(S) = V(I)$ where $I$ is the ideal of $\Omega$ generated by $S$.

*If exercise.*

Given a collection $\{ I_\alpha \}$ of ideals of $\Omega$

$$\bigcap I_\alpha = V \left( \bigcup I_\alpha \right)$$

Given $I_1, \ldots, I_n$ ideals of $\Omega$

$$V(I_1) \cup \ldots \cup V(I_n) = V(I_1, \ldots, I_n)$$

**Thm** Hilbert Basis Theorem

Every ideal of $\Omega = K[x_1, \ldots, x_n]$ is finitely generated.

*If deferred.*

**Corollary** Every algebraic subset of $K^n$ has the form $V(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in \Omega$.

**Note** Let $R$ be a commutative ring in which every ideal is finitely generated.

Call $R$ NOETHERIAN.
Let $R$ be a commutative ring with 1.

Set $\operatorname{Max}(R)$ to be the set of maximal ideals of $R$.

If $I$ is any ideal of $R$, set $V(I) = \{ P \in \operatorname{Max}(R) \mid P \supseteq I \}$

Prop. The sets $V(I)$ for ideals $I$ of $R$ taken as the closed sets, form a topology on $\operatorname{Max}(R)$.

First, $V(0) = \emptyset$ and $V(R) = \mathcal{O}$. Next, the intersection

$$\bigcap_{i} V(I_i) = V(\bigcup_{i} I_i) \quad (\text{check})$$

for collections $\{ I_i \}$ of ideals of $R$.

Consider ideals $I, J$ of $R$. Let $P$ be a maximal ideal of $R$. Recall that $P$ is thus a prime ideal. We have shown if $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

Therefore, $V(IJ) = V(I) \cup V(J)$

Consequently, an easy induction shows

$$V(I_1) \cup \ldots \cup V(I_n) = V(I_1 \ldots I_n) \quad \square$$

Remarks. We call this topology the Zariski topology. We call $\operatorname{Max}(R)$ with this topology as the MAXIMAL SPECTRUM of $R$, $\operatorname{Spec}R$.

Let $\operatorname{Spec}R$ be set of prime ideals of $R$, can we put the Zariski topology?

Why $\operatorname{Spec}R$ not $\operatorname{Max}(R)$?

$\operatorname{Spec}R$ is functorial, $\operatorname{Max}(R)$ is not.
Hilbert's Nullstellensatz

Let $k$ be an algebraically closed field. Then the maximal ideals of $O = k[x_1, \ldots, x_n]$ are precisely the ideals of the form $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ for $\lambda_1, \ldots, \lambda_n \in k$. Over $\mathbb{C}$ this is no longer so.

Proof: Elsewhere \(\Box\)

Remarks

1. For any field $k$ (i.e., not necessarily algebraically closed), the ideals of the form $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ are maximal, but this could be otherwise.

To see why, note that $(x_1 - \lambda_1, \ldots, x_n - \lambda_n)$ is the kernel of the homomorphism:

$$k[x_1, \ldots, x_n] \rightarrow k$$

$$x_1 \mapsto \lambda_1$$

$$\vdots$$

$$x_n \mapsto \lambda_n$$

$$f \mapsto f(\lambda_1, \ldots, \lambda_n)$$

Kernel of homomorphism onto a field, so must be maximal.

2. For $k$ algebraically closed, can identify $\text{Max } O$ with $k^n$ via

$$(x_1 - \lambda_1, \ldots, x_n - \lambda_n) \Leftrightarrow (\lambda_1, \ldots, \lambda_n)$$

This identification is a homeomorphism under the Zariski topologies given.

3. In $k[x_1]$, $(x^{k+1})$ is a maximal ideal not of form $(x - \lambda)$ for any $\lambda \in k$, so maximal spectrum is larger.
Def. An algebraic subset $V$ of $K^n$ is irreducible if for all algebraic subsets $X, Y$ of $K^n$

$$V = X \cup Y \implies V = X \lor V = Y$$

Prop. Let $I$ be an ideal of $\mathcal{O}$. Then $V(I)$ is irreducible if $I$ is prime.

Def. For a topological space $X$ we can define the dimension to be $n$ when the maximal possible "length" of a series of inclusions of irreducible subspaces

$$\emptyset = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$$

is $n$.

Can be reinterpreted for $K^n$ in terms of chains of prime ideals (also meaningful for closed subsets)

In particular, dimension of $K^n$ is $n$. 