Def. A a set and for each \( a \in A \) let \( M_a \) be a left \( R \)-module.

i) The direct product

\[
\prod_{a \in A} M_a
\]

is the set of \( a \)-tuples

\[
\{(m_a)_{a \in A} \mid m_a \in M_a\}
\]

with addition

\[
(m_a)_{a \in A} + (m'_a)_{a \in A} = (m_a + m'_a)_{a \in A}
\]

and scalar product

\[
R \cdot (m_a)_{a \in A} = (r \cdot m_a)_{a \in A}
\]

(check \( R \)-module axioms).

ii) The direct sum

\[
\bigoplus_{a \in A} M_a
\]

is the \( R \)-submodule of \( \prod_{a \in A} M_a \) where all but finitely many terms are zero.

iii) If \( A \) finite, they are isomorphic.

iv) Let \( M \) be an \( R \)-module. Suppose \( M = M_a \) for all \( a \in A \). Then

\[
M_A := \bigoplus_{a \in A} M_a
\]
Prop Let $A$ be a set and let $E = R^A$ (where $R$ means $\mathbb{R}$). Left $R$-module $R^A$ let $A' = \{(a_x) \in A : r_a = 0 \text{ for all } a \in A, r_a = 1 \text{ for all } x \in A\}$. Then $E$ is a free $R$-module with basis $A'$. In particular, there is an $R$-module isomorphism $F(A) \rightarrow E$ that bijectively maps $A$ onto $A'$.

Remarks

1) Direct products and sums of Abelian groups "correspond exactly" to direct products and sums of $\mathbb{Z}$-modules.

2) Direct sums are also referred to as coproducts.

3) See last semester for example on $\mathbb{Z}$ embedding into $\prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$; a direct product of finite abelian groups.

INTERNAL DIRECT SUMS

$N_1, \ldots, N_k$ $R$-modules. Then

$$(n_1, \ldots, n_k) = (n'_1, \ldots, n'_k) \text{ in } N_1 \times \ldots \times N_k$$

\[\iff\]

$n_1 = n'_1$, $n_2 = n'_2$, \ldots, $n_k = n'_k$.

Similarly for infinite direct products and sums.

Lemma Let $M$ be an $R$-module and $N_1, \ldots, N_k$ submodules of $M$. Then there is a well-defined surjective homomorphism

$$N_1 \times N_2 \times \ldots \times N_k \rightarrow N_1 + \ldots + N_k \subseteq M$$

$$(m_1, m_2, \ldots, m_k) \rightarrow m_1 + \ldots + m_k$$
Prop. Let $N_1, \ldots, N_t$ be submodules of $M$. Then the map

$$\begin{align*}
N_1 \times \cdots \times N_t & \rightarrow N_1 + N_2 + \cdots + N_t \subseteq M \\
(n_1, \ldots, n_t) & \mapsto n_1 + \cdots + n_t
\end{align*}$$

is an isomorphism

$\iff$

$$N_j \cap (N_1 + N_2 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_t) = 0$$

for all $1 \leq j \leq t$ (N.B. this is a stronger condition than pairwise trivial intersection).

If already known $\Phi$ is an epimorphism. Now suppose

$$(n_1, \ldots, n_t) \in \ker \Phi$$

To prove $\Rightarrow$ : Assume $\Phi$ is an isomorphism and $N_j \cap (N_1 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_t)$ nontrivial for some $j$. Then there exists $0 \neq n_j \in N_j$ such that

$$n_j = (n_1 + n_2 + \cdots + n_{j-1} + n_{j+1} + \cdots + n_t) = 0$$

for some $n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_t \in$ respective sets.

Then $$(n_1, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_t) \in \ker \Phi$$ hence not iso.

$\Leftarrow$ Suppose $\Phi$ not isomorphism. Take $(n_1, \ldots, n_t) \in \ker \Phi$ non zero element but then $n_1 + \cdots + n_t = 0$ so $-n_j = n_1 + \cdots + n_t \neq 0$ so have nontrivial intersection.

\[\text{Notation}\]

$N_1, \ldots, N_t$ submodules such that intersection condition $N_j \cap (N_1 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_t)$ we write $N_1 \oplus \cdots \oplus N_t$ to denote $N_1 + \cdots + N_t$. 

\[\]
Throughout, let $R$ be a ring with identity 1.

**THE BASIC CONSTRUCTION**

Let $M$ be a Right $R$-module.
Let $N$ be a Left $R$-module.

Let $U(M, N)$ be the free $\mathbb{Z}$-module (or free abelian group) on the set $M \times N$.

Let $V(M, N)$ be the $\mathbb{Z}$-submodule of $U(M, N)$ generated by all elements of the form:

- $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$
- $(m, r, n) - (m, r, n)$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.

(Note, can view $MN$ as subset of $U(M, N)$)

We write

$$M \otimes_R N := \frac{U(M, N)}{V(M, N)}$$

which we call the **TENSOR PRODUCT (OVER $R$)** of $M$ and $N$. 
Note. In general, no $R$-module structure has been defined for $M \otimes_R N$.

(If $R$ is a $\mathbb{Z}$-module).

We write the image of $(m, n) \in M \times N \subseteq \text{UL}(M, N)$ in $M \otimes_R N$ as $m \otimes n$

(read aloud as "in tensor $n".)

In general, this, the elements of $M \otimes_R N$ have the form
\[ \sum_{i=1}^{n} m_i \otimes n_i \]

Elements in $M \otimes_R N$ of form $m \otimes n$ are called \textit{pure = simple sum}.

Some elementary calculations

1) $0 \otimes 0 = (0 + 0) \otimes 0 = 0 \otimes 0 + 0 \otimes 0$

$\therefore 0 \otimes 0$ is the zero element of $M \otimes_R N$

2) $a \cdot r \otimes b = a \otimes r \cdot b$

3) $a \otimes 0 = a \otimes 0 \cdot 0 = a \cdot 0 \otimes 0 = 0 \otimes 0$

$0 \otimes b = 0 \cdot 0 \otimes b = 0 \otimes 0 \cdot b = 0 \otimes 0$

\underline{Example}

$R = \mathbb{Z}, M = \mathbb{Q}, N = \mathbb{Z}/n\mathbb{Z}, n \in \mathbb{Z}_{>0}$

Consider $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = M \otimes_R N$

Choose arbitrary simple tensor $a \otimes b$.

Then $a \otimes b = (\frac{a}{n}) \otimes (\frac{b}{n}) = \frac{a}{n} \otimes \frac{b}{n} = (\frac{a}{n}) \otimes 0 = 0 \otimes 0$

Thus $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$
Def. Let $M$ be a right $R$-module, $N$ a left $R$-module and $L$ an abelian group. A function

$$\Phi : M \times N \rightarrow L$$

is called **$R$-balanced** if

$$\Phi(m_1 + m_2, n) = \Phi(m_1, n) + \Phi(m_2, n)$$

$$\Phi(m, n_1 + n_2) = \Phi(m, n_1) + \Phi(m, n_2)$$

$$\Phi(m, rn) = \Phi(m, r)n$$

Example:

1) The multiplication map

$$R \times R \rightarrow R$$

$$(a, b) \mapsto ab$$

is $R$-balanced.

2) If $I$ is a right ideal of $R$, $J$ a left ideal

$$I \times J \rightarrow R$$

$$(i, j) \mapsto ij$$

is $R$-balanced.

Next time:

The tensor product is the universal object associated with $R$-balanced maps — $R$-balanced maps factor through tensor products.

$$M \times N \rightarrow M \otimes_R N \rightarrow L$$