§13.2 Algebraic Extensions (cont)

Let $K/F$, $a \in K$, algebraic over $F$ if $\exists f \in F[x] \neq 0$ algebraic extension $[K:F]$.

Thm $[K:F] < \infty \iff K$ finitely generated by algebraic elements. $K = F(a_1, \ldots, a_n)$, $a_i$ algebraic over $F$.

Eg $\alpha/\beta$ algebraic if $\alpha, \beta$ are.

$m_{\alpha, F}(X)$ minimum monic polynomial vanishing at $\alpha$.

Def Let $\alpha \in K$ be algebraic over $F$. The degree of $m_{\alpha, F}(X)$ is called the DEGREE of $\alpha$.

Lemma Let $\alpha \in K$ be a root of some $f(x) \in F[x]$.

Then the degree of $\alpha \leq \deg f(x)$.

Pf From last time we know that $m_{\alpha, F}(X)$.

From last time $m_{\alpha, F}(X) \mid f(x)$.$\square$

Prop Suppose $K = F(\alpha_1, \ldots, \alpha_m)$ and each $\alpha_i$ is algebraic over $F$ of degree $n_i$. Then $[K:F] = n_1 \cdots n_m$. 
If set $F_i = F(\alpha_1, \ldots, \alpha_i)$ for $1 \leq i \leq n$ and set $F_0 = F_1$ for each

$$F_i = F_{i-1} \cup \{\alpha_i\}$$

the degree of $\alpha_i$ over $F_{i-1}$ is the degree of $\alpha_i$ over $F_1 = F$.

So result follows.

$[K:F] = [F_n:F_{n-1}][F_{n-1}:F_{n-2}] \cdots [F_1:F_0] \\ \leq n_1 \cdots n_r$

Corollary

Let $\alpha, \beta \in K$, $\beta \neq 0$. Suppose $\alpha, \beta$ algebraic over $F$. Then $\alpha + \beta$, $\alpha \beta$, $\alpha / \beta$ algebraic over $F$.

Corollary

Let $F \subseteq L$ be a field extension.

Let $K = \{\alpha \in L \mid \alpha$ is algebraic over $F\}$.

Then $K$ is a field extension of $F$.

If $K$ is closed under field operations, so is a subfield of $L$.

Ex. Set $\overline{Q} = \{\alpha \in C \mid \alpha$ algebraic over $Q\}$

not a finite extension, since, e.g. $\sqrt{2} \in \overline{Q}$, but are of arbitrarily high degree, since $x^n - 2$ is irreducible for all $n$ (by Eisenstein), and since $[Q(\sqrt[2]{2}) : Q] = [Q(\sqrt[2]{2}) : Q] [Q(\sqrt[2]{2}) : Q]$

so that $[Q(\sqrt[2]{2}) : Q] = \infty$ by before
Note that \( \overline{\mathbb{Q}} \) is countable, \( \overline{\mathbb{Q}} \subseteq \overline{\mathbb{C}} \).

**Theorem.** \( F \subseteq K \subseteq L \) field extensions. If \( K \) is algebraic over \( F \) and \( L \) is algebraic over \( K \), then \( L \) is algebraic over \( F \).

*Proof.* Choose \( a \in L \). Then
\[
a_n a^n + \ldots + a_0 = 0
\]
for some positive \( n \) and \( a_0, \ldots, a_n \in K \). Consider \( F(a_0, \ldots, a_n) \). Each of the \( a_0, \ldots, a_n \) is algebraic over \( F \).

\[
[F(a_0, \ldots, a_n) : F] < \infty \text{ by above}
\]

Also, \( [F(\alpha, a_0, \ldots, a_n) : F(a_0, \ldots, a_n)] < \infty \)

since \( \alpha \) is algebraic over \( F(a_0, \ldots, a_n) \).

\[
[F(\alpha) : F] < \infty \text{ since } F(\alpha) \subseteq F(\alpha, a_0, \ldots, a_n)
\]
so \( \alpha \) is algebraic over \( F \).

\[\Box\]

**Homework for Thursday: posted**

- §13.3 Classical Straightedge & Compass

Consider the following (idealized) constructions in the plane:

1. Drawing line segment connecting two points
2. Finding the intersection of two segments
3. Drawing a circle of given center, radius
4. Finding intersections of a circle with a line or circle
We also have a designated, fixed length 1.

**Def:** A real number that can be realized as a length using above constructions is CONSTRUCTIBLE.

**Prop 1.** If \( a, b \in \mathbb{R} \) are constructible, then so are:
- \( a + b \)
- \( a - b \)
- \( ab \)
- \( a/b \)
- \( \sqrt{a} \)

In particular, the set of constructible numbers is a subfield of \( \mathbb{R} \) containing \( \mathbb{Q} \) and closed under square roots.

**Prop 2.** Let \( \alpha \in \mathbb{R} \) be constructible, then \( [\mathbb{Q}(\alpha): \mathbb{Q}] = k^t \) for some nonnegative integer \( t \).

**Pf:** Constructible numbers arise from degree 2 or fewer extensions.

---

**The Classical Unsolved Questions**

I. Doubling the cube (volume of a cube 2

II. Trisect an angle

III. Squaring the circle (i.e., construct a square of same area)

**Why**

I. \( \sqrt[3]{2} \) not constructible, since \( [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 \neq k^t \)

II. Would imply the root of irreducible \( x^3 - 3x - 1 \) is constructible.

III. Would imply \( \pi \) is algebraic.
§13.4 Splitting Fields and Algebraic Closure

Throughout, let $F$ be a field.

**Definition.** Let $f(x) \in F[x]$, not scalar, and let $K$ be an extension field of $F$.

Say $f(x)$ Splits Completely in $K[x]$ if there exist $\lambda$, $\alpha_1$, $\ldots$, $\alpha_n \in K$ s.t.

$$f(x) = \lambda(x - \alpha_1) \cdots (x - \alpha_n)$$

Say that $K$ is a Splitting Field over $F$ for $f(x)$ if $f(x)$ splits in $K[x]$ but does not split over any proper subfield of $K$ containing $F$.

**Theorem.** Let $f(x) \in F[x]$ not scalar. Then there exists an extension field of $F$ that is a splitting field for $f(x)$ over $F$.

**Proof.** First, if $\deg f(x) = 1$, then $F$ contains all roots of $f(x) = ax + b$.

Assume now $\deg(f(x)) = n > 1$. Then we know there exists a field extension $F(\alpha)$ of $F$ with $\alpha$ a root of $f(x)$. Therefore $f(x) = (x - \alpha)g(x)$ in $(F(\alpha))[x]$ for some $g(x) \in (F(\alpha))[x]$. Note $\deg g(x) = n - 1$ so by induction...
So there is a field extension $L$ of $F(a)$ in which $g(x)$ splits completely.

\[ f(x) \text{ splits completely in } L. \]

Now take $K$ the intersection of all subfields of $L$ containing $F$ in which $f(x)$ splits completely.

\[ eg \quad x^3 - 2 \]