Question  \[ H, K \leq G \quad HK \cong H \times K \quad \Rightarrow \quad H \cap K = 1 \]

Non-zero torsion free abelian group is free?

Recall a an abelian group, additive
\[ tA = \{ a \in A \mid na = 0 \quad \text{for some } n = 1, 2, \ldots\} \]

Then non-zero finitely generated torsion free group \( A \)
is free.

\( \text{Def} \) A free abelian group rank \( r \) if \( A \cong \mathbb{Z} \times \mathbb{Z} \times \ldots \)

Lemma A an abelian group. Then \( A/tA \) is torsion free.

Proof Choose \( a + tA \in A/tA \) if
\[ n(a + tA) = 0 \quad \text{then } \quad na \in tA \quad \text{so } \quad a + tA = 0 + tA \]
\[ \therefore \quad t(\frac{A}{tA}) = \langle 0 \rangle \quad \text{in } \frac{A}{tA} \quad \Box \]

Note \( |A| = n < \infty \) must not be torsion since \( na = 0 \) back

Lemma A finitely generated torsion abelian group must be finite.
If \( A = \langle a_1, \ldots, a_k \rangle \) be additive abelian group, show either
that \( A = \{ n_1 a_1 + n_2 a_2 + \ldots + n_k a_k \mid n_1, \ldots, n_k \in \mathbb{Z} \} \)
but \( A \) is torsion, so \( ma = 0 = mn_1 a_1 + mn_2 a_2 + \ldots + mn_k a_k \).
A torsion, so for each \( a_i \in M \), \( m_i a_i = 0 \)

\[
\text{thus } M = \{{} \text{im}_i \}_{i=1}^3
\]

then exist at most \( m \) may distinct sums, since

\[
A = \{ \sum n_i a_i : -M \leq n_1, n_2, \ldots, n_r \leq M \}
\]

\[
|A| < \infty \quad \Box
\]

Then let \( A \) be finitely generated abelian group

Then \( A \) is isomorphic to a direct product of an abelian group of finite rank and a finite group

i.e. \( |B| < \infty \), \( A \cong \mathbb{Z}^r \times B \)

pf: Consider projection \( \pi : A \to A/\tau A \)

Then \( A/\tau A \) is finitely generated \& torsion free (from before)

\[
\therefore \pi(A) \text{ is free by earlier}
\]

Also \( \ker \pi = \tau A \) \& again by earlier

\[
A \cong \pi(A) \times \tau A
\]

Remains to show \( \tau A \) finite, but suffices to show \( \tau A \) is finitely generated

Note, since \( A \cong \pi(A) \times \tau A \), and since there is a surjection

\[
\pi(A) \times \tau A \longrightarrow \tau A
\]

\[ (u, v) \longrightarrow v \]

we see \( \tau A \) is an image of finitely generated \& \( \tau A \) is finitely generated
Cor. A finitely generated abelian group, then
\[ A \cong \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}/m_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/m_n \mathbb{Z} \]
for some \( r \geq 0 \) and \( m_1, \ldots, m_n \in \mathbb{Z}^+ \)

This is the core of the fundamental theorem of finitely generated abelian groups.

\[ \text{FREE Groups of $\mathbb{Z}$} \]

\[ \langle r, s \mid r^n = s^2 = 1, \ rs = sr^{-1} \rangle \]

denote \( S \) be a nonempty set.

For all positive integers \( m \) and all \( S_1, \ldots, S_m \in S \),
we may write the FORMAL PRODUCT

\[ S_1 S_2 \cdots S_m \]

Then the formal products of \( S \) are called WORDS in the elements of \( S \).

In general, a word in \( S \) is not an element of \( S \).

Define \( S^{-1} \) to be the set of "formal inverses", \( s^{-1} \), for \( s \in S \).

One formally, \( S^{-1} \) is a set disjoint to \( S \) with bijection \( S \leftrightarrow S^{-1} \).
Now let $X = S \cup S^{-1}$

We consider words now in $X$

If $u = x_1 \ldots x_m$

$$v = x_{m+1} \ldots x_n$$

are words in $X$ for $x_1 \ldots x_n \in X$

define $uv$ to be $x_1 \ldots x_m x_{m+1} \ldots x_n$, a word in $X$,
called the **concatenation** of $u$ and $v$.

Two words $u, v$ in $X$ are **directly equivalent** provided for some $x \in X$ that one of the following four cases occurs:

1. $u = v$ (i.e., *otherwise identical*)

2. $u = xx^{-1}v$

3. $u = vxx^{-1}$

4. $u = xx^{-1}s$ and $v = rs$

for some words $r, s$ in $X$ or

$v = xx^{-1}s$ and $u = rs$

Two words $u, v$ in $X$ are **equivalent** provided

there is a finite sequence of words in $X$

$$u = w_1, w_2, \ldots, w_n = v$$

such that $w_i$ directly equivalent to $w_{i+1}$

Exercise - this is an equivalence relation.
Lemma  \[ u, u', v, v' \] are words in \( X \). Suppose \( u = u' \) and \( v = v' \).

Lemma \( x, y \in X, \; \) then \( xx' \sim yy' \) \( xx' \sim xx' yy' \sim yy' \) by above rules.

Let \( I \) denote the equivalence class of \( xx' \).

Lemma \( u, u', v, v' \) be words in \( X \).

Suppose \( u = u' \) and \( v = v' \) then \( u v = u v' \).

If we first show that \( u v \sim u v' \) by definition, there is a sequence \( u = w_1, w_2, ..., w_n = v \) where each \( w_i \) is directly equivalent to \( w_{i+1} \).

By an induction omitted, we can reduce to the case where \( u \) is directly equivalent to \( u' \).

\[ u' = xx' u \text{ or } u' = u xx' \]

\[ u' = xx' s \text{ for } u = rs \]

\[ u' = rs \text{ for } u = rs xx' \]

for some \( x \in X, \text{ } r, s \text{ words in } X \).

In any of these cases, \( u v \sim u v' \) (check).

Similarly \( u v \sim u v' \) so \( u v = u v' \).

So now, for a word \( u \) in \( X \), let \( [u] \) be the equivalence class of \( u \). Given words \( u, v \) in \( X \), define \( [u][v] = [uv] \), which is well-defined by the lemma.

Also, for all words \( [u][1] = u \) \( [xx^{-1}] = [xx^{-1}] = [xx^{-1} u] = [u]^{-1} \).
Let \( F(S) := \{ [u] \mid u \text{ a word in } X \} \)

Theorem \( F(S) \) is a group under concatenation, with identity \( 1 \) as above.

Proof. Well-definedness from above:
\[
u = x_1 \ldots x_k, \quad v = x_{k+1} \ldots x_m, \quad w = x_{m+1} \ldots x_n
\]

Then:
\[
([u][v])[w] = [u][v][w] = [x_1 \ldots x_m][w] = [x_1 \ldots x_n]
\]

\[
[u][v][w] = [u][x_{k+1} \ldots x_n] = [x_1 \ldots x_n]
\]

so associativity.

We checked that \( 1 \) is an identity element.

Closure by definition.

Lastly, given \( x_1 \ldots x_m \in X \), check that:
\[
[x_1 \ldots x_m][x_1^{-1} x_1^{-1} \ldots x_1^{-1}] = [1]
\]

\[
[x_1 \ldots x_n x_n^{-1} \ldots x_1^{-1}] = [x_1 \ldots x_{n-1}][x_n x_n^{-1}][x_{n-1}^{-1} \ldots x_1^{-1}]
\]

\[
= [x_1 \ldots x_{n-1}]1[x_{n-1}^{-1} \ldots x_1^{-1}] \quad \text{by induction} \quad \square
\]

We call \( F(S) \) the FREE GROUP on \( S \).

Now define a map \( S \overset{\iota}{\rightarrow} F(S) \) which sends \( s \mapsto [s] \).

Lemma \( S \overset{\iota}{\rightarrow} F(S) \) is injective.

It follows by def'n of equivalence. \( \square \)