SOLUTION TO PROBLEM #1572
PROPOSED BY
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Problem #1572: [P] Let \( b_0 = 1 \) and \( b_1 \) satisfy \( 0 < b_1 < 1 \). For \( n \geq 1 \), define \( b_{n+1} \) by

\[
b_{n+1} = \frac{2b_nb_{n-1} - b_1^2}{3b_{n-1} - 2b_n}.
\]

Show that \( (b_n)_{n \geq 0} \) converges, and compute its limit in terms of \( b_1 \).

Solution: From the definition of \( b_{n+1} \) and since \( b_0 = 1 \), it follows that

\[
b_{n+1} = b_1 \frac{2 - b_1}{3 - 2b_1} \frac{4 - 3b_1}{5 - 4b_1} \cdots \frac{2n - (2n - 1)b_1}{2n + 1 - 2nb_1} = b_1 \prod_{k=1}^n \frac{2k - (2k - 1)b_1}{2k + 1 - 2kb_1},
\]

which is easily proven by induction on \( n \). Letting \( \beta := 1/(2 - 2b_1) > 1/2 \), and using binomial coefficients we may rewrite this as follows:

\[
b_{n+1} = b_1 \prod_{k=1}^n \left(1 - \frac{1}{2k + 2\beta}\right) = \frac{2\beta - 1}{2\beta\binom{\frac{\beta}{2}}{\beta}} \cdot \frac{\binom{2n + 2\beta}{n + \beta}}{4^n}.
\]

Using Stirling’s approximation for \( n! \) as \( n^e e^{-n} \sqrt{2\pi n} \), we get for any \( \beta > 1/2 \)

\[
\lim_{n \to \infty} b_{n+1} = (\text{constant}) \cdot \lim_{n \to \infty} \frac{1}{\sqrt{n + \beta}} = 0. \square
\]

References: