SOLUTION TO PROBLEM #11909

Problem #11909. Proposed by H. Ohtsuka, Japan. Prove that for every positive integer \( m \), there exists a polynomial \( P_m \) in two variables, with integer coefficients, such that for all integers \( n \) and \( r \) with \( 0 \leq r \leq n \),
\[
\sum_{k=-r}^{r} \binom{n}{r+k} \binom{n}{r-k} k^{2m} = \frac{P_m(n, r)}{\prod_{j=1}^{m} (2n - 2j + 1) 2r}.
\]

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. From the Vandermonde-Chu identity, we get
\[
\sum_{i=0}^{r} \binom{n}{r+i} \binom{n}{r-i} = \sum_{r-i}^{r} \binom{n}{r} \binom{n}{r-i} = \binom{2n}{2r}.
\]
Next, observe that \( k^{2m} \) can be expanded in different bases as follows
\[
k^{2m} = \sum_{i=0}^{m} \alpha_i \prod_{t=0}^{i-1} (r+k-t)(r-k-t) = \sum_{i=0}^{m} \alpha_i \frac{(r+k)!(r-k)!}{(r+k-i)!(r-k-i)!} \quad \text{for some } \alpha_i \in \mathbb{Z}[r].
\]
For example, \( k^2 = -(r+k)(r-k) + r^2 \). Then, the sum on the left-hand side of the original problem takes the form
\[
\sum_{i=0}^{m} \sum_{k=-r}^{r} \binom{n}{r+k} \binom{n}{r-k} \frac{\alpha_i (r+k)!(r-k)!}{(r+i+k)!(r+i-k)!} = \sum_{i=0}^{m} \alpha_i \frac{n!^2}{(n-i)!^2} \sum_{k=-r-i}^{r-i} \binom{n-i}{r-i+k} \binom{n-i}{r-i-k} = \sum_{i=0}^{m} \alpha_i \frac{n!^2}{(n-i)!^2} \binom{2n-2i}{2r-2i} = \binom{2n}{2r} \sum_{i=0}^{m} \alpha_i \frac{n!^2}{(n-i)!^2} \frac{(2r)!}{(2r-2i)!} \frac{(2n-2i)!}{(2n)!}.
\]
If \( 1 \leq i \leq m \), then \( \frac{n!(n-1)!}{(n-i)!^2} \) and \( \frac{(2r-1)!}{(2r-2i)!} \) are polynomials in \( n \) and \( r \) respectively, of finite degree (depending on \( m \)). The remaining contributions simplify to
\[
\frac{2n(2n-2i)!}{(2n)!} = \frac{1}{\prod_{j=1}^{m} (2n - 2j + 1)}
\]
which is a rational function with denominator a factor of \( \prod_{j=1}^{m} (2n - 2j + 1) \). The polynomial promised under the claim is then
\[
P_m(n, r) = \sum_{i=0}^{m} \alpha_i (r)^{n!(n-1)!r(2r-1)!} \frac{(2r)!}{(2r-2i)!} \frac{(2n-2i)!}{(2n)!} \prod_{j=i+1}^{m} (2n - 2j + 1).
\]
The proof is complete. □