Problem #11902. Proposed by Cornel Ioan Valean, Teremia Mare, Timis, Romania. Let \( \{x\} \) denote \( x - [x] \), the fractional part of \( x \). Prove
\[
\int_0^1 \int_0^1 \int_0^1 \left( \frac{x}{y} \right) \{ \frac{x}{y} \} \left\{ \frac{x}{z} \right\} \left\{ \frac{z}{x} \right\} dz \, dy \, dx = 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + 7\frac{\zeta(6)}{48} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{12} + \frac{\zeta(3)\zeta(4)}{12}.
\]

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA, USA. The 6 permutations of the chamber \( 0 \leq x \leq y \leq z \leq 1 \) do partition the unit cube into tetrahedra. In view of the cyclic symmetry of the integrand, we obtain exactly 2 different triple integral evaluations, each appearing thrice.

Case 1 is modeled by \( 0 \leq x \leq y \leq z \leq 1 \) and \( \{ \frac{x}{y} \} \{ \frac{y}{z} \} = \frac{x}{y} \{ \frac{y}{z} \} = \frac{x}{y} \{ \frac{z}{x} \} \). Integrate by parts: let \( u = \int_0^y (\cdots) \) and \( v' = 1 \), so \( v = \frac{y^2}{2y(\frac{y}{x})^2} \) and \( v = y \). Consequently,
\[
I_1 := \int_0^1 \int_0^z \int_0^y \left( \frac{x}{z} \right) \left( \frac{z}{y} \right) \left( \frac{z}{y} \right) \, dx \, dy \, dz = \int_0^1 \left( \int_0^y \int_0^{y^2} \left( \frac{z}{y} \right)^2 \, dz \right) \int_0^1 \frac{y^2}{2y(\frac{y}{x})^2} \, dx \, dy \, dz = \int_0^1 \int_0^z \int_0^y \frac{z^2}{2y} \left( \frac{z}{y} \right)^2 \, dx \, dy \, dz \quad \text{and hence}
\]
\[
I_{1,1} = \int_0^1 \int_0^z \int_1^\infty \frac{(w)^2}{w^4} \, dw \, dw = \frac{1}{3} \int_1^\infty \frac{(w)^2}{w^4} \, dw = \frac{1}{3} \sum_{n=1}^\infty \frac{1}{(n + 1)^3} = \frac{1}{9} \left( 3 - \zeta(2) - \zeta(3) \right) = \frac{1}{3} - \frac{\zeta(2)}{9} - \frac{\zeta(3)}{9}.
\]

Likewise, the second integral results in \( I_{1,2} := \int_0^1 \int_0^z \int_1^\infty \frac{(w)^2}{w^4} \, dw \, dw = \frac{1}{6} - \frac{\zeta(3)}{12} - \frac{\zeta(4)}{12} \).

Case 2 is modeled by \( 0 \leq z \leq x \leq y \leq 1 \) so that \( \{ \frac{x}{y} \} \{ \frac{y}{z} \} = \{ \frac{x}{y} \} \{ \frac{z}{x} \} \). Make the substitution \( w = \frac{y}{x} \) so that \( dx = -\frac{x}{y} \, dw \) (analogously, \( q = \frac{x}{y} \), \( dz = -\frac{y}{x} \, dq \) and hence
\[
I_2 := \int_0^1 \int_0^1 \int_0^y \left( \frac{x}{y} \right) \left( \frac{z}{x} \right) \, dx \, dy \, dz = \int_0^1 \int_0^y \int_0^y \frac{z}{y^2} \, dx \, dy \, dz = \int_0^1 \int_0^y \frac{z}{y^2} \, dy \, dz \cdot \int_0^1 \frac{(w)^2}{w^4} \, dw \;
\]
\[
= \frac{1}{9} \int_0^1 \int_0^y \frac{z}{y^2} \, dy \, dz \cdot \int_1^\infty \frac{(w)^2}{w^4} \, dw = \frac{1}{3} \left( 1 - \frac{\zeta(2)}{6} - \frac{\zeta(4)}{4} \right) \left( 1 - \frac{\zeta(2)}{3} - \frac{\zeta(3)}{3} \right).
\]

Combining all that we have found, the integral in the problem becomes \( I = 3I_{1,1} - 3I_{1,2} + 3I_2 \), i.e.
\[
I = 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{\zeta(2)\zeta(4)}{12} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{18} + \frac{\zeta(3)\zeta(4)}{12}.
\]

The proof is complete. \( \square \)