Problem #11899. Proposed by J. Sorel, Romania. Show that for any positive integer $n$,

\[
\sum_{k=0}^{n} \binom{2n}{k} \binom{2n+1}{k} + \sum_{k=n+1}^{2n+1} \binom{2n+1}{k-1} \binom{2n+1}{k} = \binom{4n+1}{2n} + \binom{2n}{n}^2.
\]

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Start with $A_1 := \sum_{k=0}^{n} \binom{2n}{k} \binom{2n+1}{k}$, $A_2 := \sum_{k=n+1}^{2n+1} \binom{2n+1}{k} \binom{2n+1}{k-1}$, $B_1 := \sum_{k=n+1}^{2n+1} \binom{2n}{k-1} \binom{2n+1}{k}$ and $B_2 := \sum_{k=0}^{n} \binom{2n}{k-1} \binom{2n+1}{k}$. Re-indexing gives $A_1 = B_1$, $A_2 = B_2$. The required identity is $A_1 + B_1 = 2A_1 = \binom{4n+1}{2n} + \binom{2n}{n}^2$. In view of the Vandermonde-Chu identity $A_1 + A_2 = \binom{4n+1}{2n}$, it suffices to prove that $A_1 - A_2 = A_1 - B_2 = \binom{2n}{n}^2$.

That is, $\sum_{k=0}^{n} \binom{2n+1}{k} \left[ \binom{2n}{k} - \binom{2n}{k-1} \right] = \sum_{k=0}^{n} \binom{2n+1}{k} \left[ \binom{2n}{k} - \binom{2n}{k-1} \right] = \binom{2n}{n}^2$.

It is routine to check that $\binom{2n+1}{n-k} \left[ \binom{2n}{n-k} - \binom{2n}{n-k-1} \right] = \binom{2n+1}{n-k} \frac{2k+1}{2n+1} = G(n, k) - G(n, k+1)$ where $G(n, k) = \binom{2n}{n+k}^2$. But, $\sum_{k=0}^{n} [G(n, k) - G(n, k+1)] = G(n, 0) - G(n, n+1) = G(n, 0) - \binom{2n}{n}^2$. \qed