Problem #11872. Proposed by Phu Cuong Le Van, College of Education, Hue, Vietnam. Let \( f \) be a continuous function from \([0, 1]\) to \( \mathbb{R} \) such that \( \int_0^1 f(x)dx = 0 \). Prove that for each positive integer there exists \( c \in (0, 1) \) such that

\[
 n \int_0^c f(x)dx = c^{n+1}f(c).
\]

**Proof.** Solution by Tewodros Amdeberhan, Tulane University, USA. First, we show that if \( g : [0, b] \to \mathbb{R} \) is continuous and \( \int_b^0 g(x)dx = 0 \) then \( \int_0^b xg(x)dx = 0 \) for some \( c \in (0, b) \). Suppose not. Since \( A(t) := \int_0^t xg(x)dx \) is continuous, either \( A(t) > 0 \) or \( A(t) < 0 \) for all \( t \in (0, b) \). WLOG assume \( A(t) > 0 \). Denote \( B(t) = \int_0^t g(x)dx \). Writing \( xg(x) = (xB(x))' - B(x) = B(x) + xg(x) - B(x) \), we obtain \( A(t) = tB(t) - \int_0^t B(x)dx > 0 \) for all \( t \in (0, b) \). By a limiting process, \( bB(b) - \int_0^b B(x)dx \geq 0 \) or \( \int_0^b B(x)dx \leq 0 \) (since \( B(b) = 0 \)). Define \( h : [0, b] \to \mathbb{R} \) continuous, and differentiable in \((0, b)\), by

\[
h(t) = \begin{cases} \frac{1}{t} \int_0^t B(x)dx & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}
\]

Then, \( h'(t) = \frac{1}{t^2}(tB(t) - \int_0^t B(x)dx) > 0 \) throughout \((0, b)\) (see above). By the Mean Value Theorem, \( h(b) - h(0) = h'(a)(b - 0) > 0 \) for some \( a \in (0, b) \). It follows that \( h(b) = \frac{1}{b} \int_0^b B(x)dx > 0 \), which is a contradiction. Therefore, there exists \( c \in (0, b) \) such that \( A(c) = \int_0^c xg(x)dx = 0 \).

Apply this result to \( g(x) = f(x) \) (with \( b = 1 \)) to get \( \int_0^1 xf(x)dx = 0 \), then to \( g(x) = xf(x) \) (with \( b = c_1 \)) to obtain \( \int_0^{c_1} x^2f(x)dx = 0 \), and so on. Hence \( \int_0^{c_n} f(x)dx = 0 \) for some \( c_n \in (0, 1) \). Let

\[
E(t) = \begin{cases} \frac{1}{t^n} \int_0^t x^n f(x)dx & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}
\]

The function \( E \) is continuous on \([0, c_n]\), differentiable in \((0, c_n)\) and \( E(0) = E(c_n) = 0 \). By Rolle’s Theorem, there exists \( \eta_n \in (0, c_n) \) such that \( 0 = E'(\eta_n) = \frac{n}{\eta_n^{n+1}} \int_0^{\eta_n} x^n f(x)dx + f(\eta_n) \). That means,

\[
n \int_0^{\eta_n} x^n f(x)dx = \eta_n^{n+1}f(\eta_n). \]

\( \square \).

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