Problem #11867. Proposed by George Apostolopoulos, Messolonghi, Greece. For real numbers \(a, b, c\), let
\[
f(a, b, c) = \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4}.
\]
Prove that \(f(a, b, c) + f(b, c, a) + f(c, a, b) \leq 3\).

Proof. Solution by Tewodros Amdeberhan, Tulane University, USA. From \((a - b)^2 \geq 0\), we get \(a^2 - ab + b^2 \geq ab\) and then \(3(a^3 + b^3) \geq 3ab(a + b)\). So, \(4(a^3 + b^3) \geq a^4 + b^3 + 3ab(a + b) = (a + b)^3\).

Hence \(\frac{1}{a^3 + b^3} \leq \frac{4}{(a+b)^3}\). Consequently, for the problem at hand, we obtain
\[
\sum_{cyc} \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4} = \sum_{cyc} \left(\frac{a^2(a+b)}{a^3 + b^3}\right)^{1/4} \leq \sum_{cyc} \left(\frac{4a^2(a+b)}{(a+b)^3}\right)^{1/4} = \sum_{cyc} \sqrt{\frac{2a}{a+b}}.
\]

At this point, Jensen’s inequality applied to the concave function \(f(x) = \sqrt{x}\) effectively yields
\[
\sum_{cyc} \sqrt{\frac{2a}{a+b}} = \sum_{cyc} \frac{a + c}{2(a + b + c)} \sqrt{\frac{8a(a + b + c)^2}{(a+b)(a+c)^2}} \leq \left[ \sum_{cyc} \frac{a + c}{2(a + b + c)} \frac{8a(a + b + c)^2}{(a+b)(a+c)^2}\right]^{1/4} = \left[ \sum_{cyc} \frac{4a(a + b + c)}{(a+b)(a+c)}\right]^{1/4}.
\]

It remains to show \(\sum_{cyc} \frac{4a(a+b+c)}{(a+b)(a+c)} \leq 9\). Clearing denominators, noting \(\sum_{cyc} a(b+c) = 2 \sum_{cyc} ab\), and expanding the resulting expressions, this last claim amounts to
\[
8 \sum_{cyc} ab \cdot \sum_{cyc} a \leq 9 \prod_{cyc} (a + b) \iff \sum_{cyc} c(a - b)^2 \geq 0.
\]

The above argument implicitly assumes \(a, b, c\) to be non-negative. Such is no loss of generality because for negative numbers, \(f(a, b, c)\) simply becomes smaller. The proof is complete. \(\square\)