Problem #11844. Proposed by H. Ohtsuka (Japan) and R. Tauraso (Italy). For nonnegative integers \( m \) and \( n \), prove
\[
\sum_{k=0}^{n}(m-2k)\binom{m}{k}^3 = (m-n)\binom{m}{n}\sum_{j=0}^{m-1} \binom{j}{n}(m-n-1).
\]

Proof. Solution by Tewodros Amdeberhan, Tulane University, and Shalosh B. Ekhad, USA. Let \( F_1(m,k) := (m-2k)\binom{m}{k}^3 \) and \( F_2(m,j) := (m-n)\binom{m}{n}\binom{j}{m-n-1} \). Notice that the assertion amounts to \( f_1(m) := \sum_{k=0}^{n} F_1(m,k) = \sum_{j=0}^{m-1} F_2(m,j) := f_2(m) \). Zeilberger’s algorithm generates the WZ-mates \( G_1(m,k) = (2m-k+2)\binom{m}{m-k}^3 \) and \( G_2(m,j) = (n+1)\binom{m}{n}\binom{j}{j+n} \) so that
\[
F_1(m+1,k)+F_1(m,k) = G_1(m+1,k) - G_1(m,k) \quad \quad F_2(m+1,j)+F_2(m,j) = G_2(m+1,j) - G_2(m,j).
\]

Summing over \( k \) and \( j \), respectively and observing telescoping properties with \( G_1, G_2 \), we find that
\[
\sum_{k=0}^{n} F_1(m+1,k) + \sum_{k=0}^{m-1} F_1(m,k) = \sum_{k=0}^{n} G_1(m,k+1) - \sum_{k=0}^{m-1} G_1(m,k+1) = (2m-n+1)\binom{m}{n},
\]
\[
\sum_{j=0}^{m-1} F_2(m+1,j) + \sum_{j=0}^{m-1} F_2(m,j) = \sum_{j=0}^{m-1} G_2(m,j+1) - \sum_{j=0}^{m-1} G_2(m,j) = m\binom{m-1}{n}\binom{m}{n}.
\]

The first of these equations offers the recurrence \( f_1(m+1) + f_1(m) = (2m-n+1)\binom{m}{n} \). After adding the term \( f_2(m+1,m) \) to both sides of the second equation, we are lead to \( f_2(m+1) + f_2(m) = m\binom{m-1}{n}\binom{m}{n} + (m+1-n)\binom{m+1}{n} - (2m-n+1)\binom{m}{n} \). The two functions \( f_1 \) and \( f_2 \) satisfy the same linear recurrence. Note: when the suppressed variable \( n \geq m \), \( f_2(m) = 0 \) trivially, while \( f_1(m) = \sum_{k=0}^{m}(m-k)\binom{m}{k}^3 - \sum_{k=0}^{m} k\binom{m}{k}^3 = 0 \). Checking \( f_1(0) = f_2(0) = 0 \) match, the identity \( f_1(m) = f_2(m) \) follows. □