Problem #11843. Proposed by Mihali Bencze, Bucharest, Romania. Let \( n \) and \( k \) be positive integers, and let \( x_j \geq 1 \) for \( 1 \leq j \leq n \). Let \( y = \prod_{i=1}^{n} x_i \). Show that

\[
\sum_{i=1}^{n} \frac{1}{1 + x_i} \geq \sum_{j=1}^{n} \frac{1}{1 + (x_j^{k-1} y)^{1/(n+k-1)}}.
\]

Proof. Solution by Tewodros Amdeberhan, Tulane University, USA. Recall Jensen’s inequality for convex functions: \( f \left( \sum_{i=1}^{n} \lambda_i a_i \right) \geq \sum_{i=1}^{n} \lambda_i f(a_i) \) for positive numbers \( \lambda_j \). The function \( f(x) = \frac{1}{1+e^x} \) is convex since \( f''(x) = e^x(1+e^x)^2 > 0 \) whenever \( x \in (0, \infty) \). Denote \( \lambda = (\lambda_1, \ldots, \lambda_n) \). Apply Jensen’s inequality to \( f \) repeatedly \( n \) times: for \( \lambda = (k, 1, 1, \ldots, 1) \); for \( \lambda = (1, k, 1, \ldots, 1) \); and so on, until \( \lambda = (1, 1, 1, \ldots, k) \). In all cases, \( \sum_i \lambda_i = n + k - 1 \). These lead to a system of \( n \) inequalities; for wit, the first and the last inequalities appear respectively as

\[
\frac{1}{n+k-1} \left[ \frac{k}{1+e^{a_1}} + \frac{1}{1+e^{a_2}} + \cdots + \frac{1}{1+e^{a_n}} \right] \geq \frac{1}{1+e^{\frac{a_1+\cdots+a_n}{n+k-1}}},
\]

\[
\frac{1}{n+k-1} \left[ \frac{1}{1+e^{a_1}} + \frac{1}{1+e^{a_2}} + \cdots + \frac{k}{1+e^{a_n}} \right] \geq \frac{1}{1+e^{\frac{a_1+\cdots+a_n}{n+k-1}}}. 
\]

Adding all \( n \) inequalities produces

\[
\sum_{i=1}^{n} \frac{1}{1+e^{a_i}} \geq \sum_{j=1}^{n} \frac{1}{1 + (e^{(k-1)a_j} a_{j+1} + \cdots + a_n)^{1/(n+k-1)}}.
\]

Now replace \( x_i = e^{a_i} \geq 1 \). The proof is complete. □

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