SOLUTION TO PROBLEM #11815
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Problem #11815. Proposed by George Apostolopoulos, Messolonghi, Greece. Let \(x, y\) and \(z\) be positive numbers such that \(x + y + z = 3\). Prove that

\[
\frac{x^4 + x^2 + 1}{x^2 + x + 1} + \frac{y^4 + y^2 + 1}{y^2 + y + 1} + \frac{z^4 + z^2 + 1}{z^2 + z + 1} \geq 3xyz.
\]

Proof. Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, USA. The obvious reduction \(a^4 + a^2 + 1 = a^2 - a + 1\) applied to the 3 variables (with \(x + y + z = 3\)) implies

\[
\frac{x^4 + x^2 + 1}{x^2 + x + 1} + \frac{y^4 + y^2 + 1}{y^2 + y + 1} + \frac{z^4 + z^2 + 1}{z^2 + z + 1} = x^2 + y^2 + z^2 - (x + y + z) + 3 = x^2 + y^2 + z^2.
\]

The claim now takes the equivalent form \(x^2 + y^2 + z^2 \geq 3xyz\). Engaging the AGM inequality (twice):

\[
x^2 + y^2 + z^2 = (x^2 + y^2 + z^2) \left(\frac{x + y + z}{3}\right)^2 \geq 3(xyz)^{\frac{3}{2}} = 3xyz.
\]