Problem #11597. Let \( f(x) = x / \log(1 - x) \). For \( 0 < x < 1 \), show that

\[
\sum_{k=1}^{\infty} \frac{x^k (1 - x)^k}{k!} f^{(k)}(x) = -\frac{x}{2} f(x).
\]

Proof. Recall the format of the Taylor series expansion of \( f(y) \), at the number \( x \), where the series converges.

\[
f(y) = \sum_{k=0}^{\infty} \frac{(y-x)^k}{k!} f^{(k)}(x).
\]

What should be \( y \) so that \( y - x = x(1-x) \)? This simply amounts to choosing \( y = 2x - x^2 \).

The next question is, what is \( f(y) \)? Well, clearly

\[
f(2x-x^2) = \frac{2x-x^2}{\log[1-(2x-x^2)]} = \frac{2x-x^2}{\log((1-x)^2)} = \frac{2x-x^2}{2\log(1-x)}.
\]

On the other hand, splitting the summand for \( k = 0 \) results in

\[
\sum_{k=0}^{\infty} \frac{(y-x)^k}{k!} f^{(k)}(x) = \sum_{k=0}^{\infty} \frac{[x(1-x)]^k}{k!} f^{(k)}(x)
\]

\[
= \frac{x}{\log(1-x)} + \sum_{k=1}^{\infty} \frac{x^k (1-x)^k}{k!} f^{(k)}(x).
\]

This says,

\[
\frac{2x-x^2}{\log((1-x)^2)} = \frac{x}{\log(1-x)} + \sum_{k=1}^{\infty} \frac{x^k (1-x)^k}{k!} f^{(k)}(x).
\]

In other words,

\[
\sum_{k=1}^{\infty} \frac{x^k (1-x)^k}{k!} f^{(k)}(x) = \frac{2x-x^2}{\log((1-x)^2)} - \frac{x}{\log(1-x)} - \frac{-x^2}{2\log(1-x)} = -\frac{x}{2} f(x)
\]

as required. \( \square \)