SOLUTION TO PROBLEM #10739
PROPOSED BY OSCAR CIAURRI

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Proposed by Oscar Ciaurri, Logroño, Spain. Suppose that $f : [0, 1] \rightarrow R$ has a continuous second derivative with $f''(x) > 0$ on $(0, 1)$, and suppose that $f(0) = 0$. Choose $a \in (0, 1)$, such that $f'(a) < f(1)$. Show that there is a unique $b \in (a, 1)$ such that $f'(a) = \frac{f(b)}{b}$.

Solution by T. Amdeberhan, Devry Institute, North Brunswick, N.J. By the Mean Value Theorem for derivatives, there exists $c \in (0, a)$ such that $f'(c) = \frac{f(a)}{a}$. $f''(x) > 0$ implies that $f'(x)$ is 1-1 and increasing, therefore $f'(c) < f'(a)$. It follows that

\[
\frac{f(a)}{a} < f'(a).
\]

Define $g(x) := f(x) - xf'(a)$. The choice of $a$, yields $g(1) = f(1) - f'(a) > 0$. But $g(a) = f(a) - af'(a) < 0$ by (1). Since $g(x)$ is clearly continuous, the Intermediate Value Theorem asserts the existence of $b \in (a, 1)$ such that $g(b) = 0$, i.e. $f'(a) = \frac{f(b)}{b}$.

Now, since $f'(x)$ is strictly increasing and $g(x)$ is differentiable, we have $g'(x) = f''(x) - f'(a) > 0$, for $x > a$. Thus $g(x)$ is one-to-one, thereby proving the uniqueness of $b$. □

References: