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NUMERICAL SOLUTION OF SINGULAR SEMI-STABLE LYAPUNOV EQUATIONS

DANIEL B. SZYLD† AND JIEYONG ZHOU‡

Abstract. A numerical method for the least squares solution of certain large sparse singular Lyapunov equations is presented. It is shown that the solution can be obtained as the sum of three parts, each of which is a solution of an easier problem and which can be computed with a low rank approximation. Numerical tests illustrate the effectiveness of the proposed method.

Key words. Lyapunov equations, matrix equations, singular matrix, semi-stable matrix, projection methods, extended Krylov subspace methods.

AMS subject classifications. 65F10, 65F30, 15A06

1. Introduction. We are interested in the solution of algebraic Lyapunov equations of the form

\[ AX + XA^T + BB^T = 0, \]

where \( A \) is an \( n \times n \) large sparse semi-stable matrix and \( B \) is an \( n \times p \) matrix, \( p < n \). A matrix \( A \) is called stable if its eigenvalues lie in the open left half complex plane, i.e., if \( \text{Re}(\lambda(A)) < 0 \), and it is called semi-stable if most of its eigenvalues lie on the open left half complex plane, except for a few semi-simple zero eigenvalues. Semi-stability implies that \( A \) is singular and that the index of \( A \) is either zero or one, and therefore \( A \) has a group inverse \( A^+ \); see, e.g., [6]. Thus, throughout this paper we assume that \( \text{ind}(A) = 1 \). We are interested in a low-rank approximation to the solution of (1.1), which can thus be written as \( X \approx ZZ^T \), for some \( n \times q \) rectangular matrix \( Z \), \( q \ll n \).

It follows that when \( A \) semi-stable, then the Lyapunov operator

\[ L(X) = AX + XA^T \tag{1.2} \]

is singular, and we shall seek the least squares solution of (1.1). We mention that \( L(X) \) may be singular under other conditions, but here we only consider the case of singular \( A \), and moreover, only the case of \( A \) semi-stable.

Semi-stable matrices play an important role when analyzing autonomous systems [7], matrix second-order systems [3] which are related to modeling vibration [19], and certain nonlinear systems [4, 5]. In the last few years, a theory for optimal semi-stable control for linear and nonlinear dynamical systems was developed [13, 15, 20], and applied to several problems. These include those with a continuum of equilibria and have many applications in mechanical systems with rigid body modes, chemical reaction systems [10], compartmental systems [11, 12] isospectral matrix dynamical systems, and dynamical network systems [17, 18, 25]. These systems cover a broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground
transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks.

When $A$ is stable, and thus nonsingular, and $n$ is small, there is a well-known direct method for the numerical solution of the Lyapunov equation (1.1) [1]. For large $n$, Saad proposed a Galerkin projection method onto the Krylov subspace $\mathcal{K}_k(A,B)$ spanned by the columns of $[B, AB, A^2B, \ldots, A^{k-1}B]$ [24]. Another approach is to use the ADI method [2, 23]. In [27], it is proposed that the projection method be used with an extended Krylov subspace of the form $\mathcal{K}_k(A,B) + \mathcal{K}_k(A^{-1}, A^{-1}B)$, and it is shown numerically that this approach produces approximations to the solutions of (1.1) in less computational time than with ADI. In [8], a rational Krylov subspace method was employed as the approximation space in the projection method, and while the projection with extended Krylov is often cheaper, the use of rational Krylov is superior to the extended Krylov subspace when, for instance, the field of values of $A$ is very close to the imaginary axis. Thus, in most nonsingular cases, projection with an extended Krylov subspace method is expected to be an effective method. See also [21] for a competitive alternative.

In the case of semi-stable $A$, (symmetric) solutions to the Lyapunov equation (1.1) exist under certain conditions, and the least squares solution is the $\mathcal{H}_2$ optimal solution [13, 16].

When considering numerical methods in this singular case, there are very few options. In our experience, projection with $\mathcal{K}_k(A,B)$ produces a method with very slow convergence. In [26], a numerical method for a special singular Sylvester equation of the form $AX + XF + EB^T$ is developed in which the projection Galerkin is used on the space $\mathcal{K}_k(F,B) + \mathcal{K}_k(F - \sigma I)^{-1}, (F - \sigma I)^{-1}B)$, for some small $\sigma$. But note that when $\sigma$ is close to zero, $(F - \sigma I)^{-1}$ maps $B$ close to $\mathcal{N}(F)$, the null space of $F$. Thus, a similar approach for the Lyapunov equation (1.1) may not work as well for a singular coefficient matrix.

In this paper, we construct a projection method for the numerical solution of the Lyapunov equation (1.1) when $A$ is semi-stable and $r = \dim(\mathcal{N}(A))$, the dimension of the null space of $A$, is not very large. The solution of the original system is composed of three parts, and each can be computed by a low rank approximation. To that end, we decompose the independent term $BB^T$ into several parts, one is $B_1B_1^T$ which makes the Lyapunov equation (1.1) consistent. The consistent part can then be solved by projection onto a space $\mathcal{K}_k(A,B_1) + \mathcal{K}_k(A^{-1}, B_1)$, where $A_1$ is nonsingular and its action coincides with $A$ on the space orthogonal to $\mathcal{N}(A)$; see section 2.

In section 3 we consider a nonsymmetric independent term. Implementation details are given in section 4 and numerical experiments are presented in section 5.

2. A decomposition of the solution. We begin by reviewing a result on the form of the null space of the Lyapunov operator $L = L(X)$ defined in (1.2); cf. [14, Theo. 4.4.5]. We denote by $\mathcal{N}(L)$ its null space, and by $\mathcal{R}(L)$ its range.

THEOREM 2.1. Suppose that $\mathcal{N}(A) = \{\text{span}\{v_i, i = 1, 2, \ldots, r\}\}$ and $\mathcal{N}(A^T) = \{\text{span}\{w_i, i = 1, 2, \ldots, r\}\}$. Then $\mathcal{N}(L) = \{\text{span}\{v_iw_j^T, i, j = 1, 2, \ldots, r\}\}$ and $\mathcal{N}(L^T) = \{\text{span}\{w_iw_j^T, i, j = 1, 2, \ldots, r\}\}$.

We collect the orthogonal vectors of the bases of the right and left null spaces of $A$ and denote the corresponding matrices $V = [v_1, \ldots, v_r]$ and $W = [w_1, \ldots, w_r]$, respectively. Of course, when $A$ is symmetric, we have $W = V$. Let us further denote by $Q$ the orthogonal projection onto the left null space of $A$, that is,

\begin{equation}
Q = WW^T.
\end{equation}
and let $P = I - Q$. In order to solve the Lyapunov equation (1.1) with singular $A$, we decompose the matrix $B$ defining the independent term, into its components in $\mathcal{N}(A)$ and in $\mathcal{N}(A)^\perp$. To that end, let

\begin{equation}
B_1 = PB \quad \text{and} \quad B_2 = (I - P)B. \tag{2.2}
\end{equation}

Then, from Theorem 2.1 and (2.1), we have that

\[ B_2B_2^T \in \mathcal{N}(L^T) = \mathcal{R}(L)^\perp \quad \text{and} \quad B_1B_1^T \in \mathcal{R}(L). \]

As a consequence, if we substruct $B_2B_2^T$ from (1.1) we obtain a consistent linear system, and we have the following result.

**Theorem 2.2.** Let $B_2$ be defined as in (2.2). The solution of the following consistent singular equation

\begin{equation}
AX + XA^T + BB^T - B_2B_2^T = 0 \tag{2.3}
\end{equation}

is the least squares solution of (1.1), and $B_2B_2^T$ is the least squares residual.

Therefore, given $B$ and $W$, we can compute $B_1$ and $B_2$, and find the least squares solution of (1.1) by solving the consistent system (2.3). Since

\[ BB^T - B_2B_2^T = B_1B_1^T + B_1B_2^T + B_2B_1^T, \]

we need to solve the following three equations

\begin{align}
(2.4) & \quad AX + XA^T + B_1B_1^T = 0 \\
(2.5) & \quad AX + XA^T + B_1B_2^T = 0 \\
(2.6) & \quad AX + XA^T + B_2B_1^T = 0.
\end{align}

Let $X_1$, $X_2$ and $X_3$ be the solutions of (2.4), (2.5) and (2.6), respectively. We note that $X_2 = X_3^T$, and therefore we only need to solve (2.4) and (2.5) to then obtain the least squares solution $X = X_1 + X_2 + X_2^T$.

As we shall see, the solution of the consistent system (2.4) can be done appropriately modifying existing methods for the case of $A$ nonsingular. For the other two systems, we have the following result.

**Theorem 2.3.** For any fixed matrix $X_0$, the following

\[ X = VW^T X_0 VW^T + \left( \int_0^\infty e^{Au} B_1 du \right) B_2^T \]

is a solution of (2.5).

**Proof.** Consider the following ODE:

\[
\frac{dX(t)}{dt} = AX(t) + X(t)A^T + B_1B_2^T, \\
X(0) = X_0.
\]

The solution in closed form is given by

\[ X(t) = e^{At} X_0 e^{A^T t} + \int_0^t e^{Au} B_1B_2^T e^{A^T u} du. \]
Consider the following Jordan canonical form

\[ A = FRF^{-1} = F \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} F^{-1}, \]

where \( R_1 \) is an \((n - r) \times (n - r)\) nonsingular block Jordan matrix, \( F = [Q_1 \ V] \), and \( F^{-T} = [Q_2 \ W] \). Let \( t \to \infty \), then \( e^{At} \to [Q_1 \ V] \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_2^T \\ W^T \end{bmatrix} = VW^T \). To complete the proof, note that \( B^T_2 e^{AT} u = B^T_2 \) because \( R(B_2) \in \mathcal{N}(A) \).

Using \( A^+ \), the group inverse of \( A \), this theorem gives rise to the following characterization of the solution of (2.5).

**Corollary 2.4.** \( X = -A^+ B_1 B_2^T \) is a solution of (2.5).

**Proof.** Since \( R(B_1 B_2^T) \in \mathcal{R}(A) \) and \( AA^+ \) is an orthogonal projection onto \( \mathcal{R}(A) \), the corollary follows.

It follows then that if \( X_1 = ZZ^T \) is the solution of (2.4), then one of the least squares solutions of (1.1) is

\[ X = ZZ^T - A^+ B_1 B_2^T - B_2 B_1^T A^+. \]

Let us now consider a numerical method for (2.4). As we have noted, we cannot use projection with extended Krylov, because \( A \) is singular. Since the columns of \( X_1 \) all lie in \( \mathcal{R}(A) \), we can replace \( A \) in (2.4) with

\[ (2.7) \quad A_1 = A - VW^T, \]

and obtain the same solution \(^1\). Thus we can use a projection method with extended Krylov to solve \( X_1 \) by using the space \( \mathcal{K}^{ext}(A_1, B_1) = \mathcal{K}(A_1, B_1) + \mathcal{K}(A_1^{-1}, B_1) \). Moreover, we will also use the projection method with this extended Krylov subspace for the solution of (2.5), since \( A^+ B_1 \in \mathcal{R}(A) \), as we describe in the following.

Let \( V_k \) be the orthogonal basis of \( \mathcal{K}^{ext}(A_1, B_1) \) and let \( H_k = V_k^T A_1 V_k \). Then we solve the projected equations

\[ H_k Y_k^{(1)} + Y_k^{(1)} H_k^T + V_k^T B_1 B_1^T V_k = 0 \]

and

\[ H_k Y_k^{(2)} = V_k^T B_1 \]

to obtain the approximate solutions \( X_1^{(k)} = V_k Y_k^{(1)} V_k^T \) for \( X_1 \) and \( X_2^{(k)} = V_k Y_k^{(2)} B_1^T \) for \( X_2 \). Since \( H_k \) is nonsingular, this iteration is well defined and the solution is unique.

As we shall see in the numerical experiments, the time of computing \( X_2^{(k)} \) is minimal.

**3. A more general equation.** We can generalize our approach to solve a Lyapunov equation with a nonsymmetric independent term, i.e., of the form

\[ (3.1) \quad AX + XA^T + BC^T = 0. \]

\(^1\)We should mention that a shifted matrix of the form \( A - \sigma I \), with \( \sigma > 0 \) has been used, e.g., in [26] to define an appropriate extended Krylov space. We experimented with this approach for our problem, and it appeared to be extremely slow.
Here, we just discuss the case that $A$ is symmetric and $\dim(\mathcal{N}(A)) = 1$. More general cases follow in a similar way.

Let $\mathcal{N}(A) = \text{span}\{v\}$ and $P = I - vv^T$. Let $C_1 = PC \in \mathcal{R}(A)$ and $C_2 = (I - P)C \in \mathcal{N}(A)$, then we have the following result.

**Proposition 3.1.** If $\mathcal{N}(A) = \text{span}\{v\}$, then

$$AX + XA^T + BC^T - B_2C_2^T = 0$$

(3.2)

is consistent and

$$B_2C_2^T = (I - P)B((I - P)B)^T = (vv^T)Bad{(vv^T)C)^T}$$

is the least squares residual of (3.1).

Since

$$BC^T - B_2C_2^T = B_1C_1^T + B_1C_2^T + B_2C_1^T,$$

we need to solve the following three equations

(3.3) $AX + XA^T + B_1C_1^T = 0$

(3.4) $AX + XA^T + B_1C_2^T = 0$

(3.5) $AX + XA^T + B_2C_1^T = 0$

To obtain the solution of (3.2). This solution is a least squares solution of (3.1). For the solutions of (3.4) and (3.5), we have following result.

**Proposition 3.2.** $X = -A^+B_1C_2^T$ is a solution of (3.4) and $X = -B_2C_1^T A^+$ is a solution of (3.5).

Since $A^+B_1$ and $A^+C_1$ belong to $\mathcal{R}(A)$, we can use the nonsingular matrix $A_1 = A - vv^T$ as before, and the projection method on the sum of two extended Krylov subspaces $\mathcal{K}^{ext}(A_1, B_1) + \mathcal{K}^{ext}(A_1, C_1)$ to compute the solutions of (3.3)--(3.5). The sum of these three solutions is the solution of (3.1).

**4. Implementation details.** In this section we present some implementation details of our proposed method. For simplicity of exposition, we describe them for the symmetric case, with some comments about the nonsymmetric case, where appropriate.

One important component of our proposed method is the computation of a basis of $\mathcal{N}(A)$. We follow the approach advocated in [9], where a backward stable LU factorization with partial pivoting of $A$ is used. Thus, if $PA = LU$, with $P$ the appropriate permutation matrix, since $\mathcal{N}(U) = \mathcal{N}(A)$, the former is computed, and this is done via (symmetric) inverse iteration. As is well known, inverse iteration for singular matrices may break down. The fix for this is to use a small shift $\sigma$ (smaller than half the gap to the smallest nonzero eigenvalue), and use inverse iteration on $U - \sigma I$. In fact, we do this in our Example 5.3, with $\sigma = 10^{-10}$. In our experience, the convergence of inverse iteration, both for the one-dimensional null space, and for the multidimensional case (when subspace iteration is used) converges very fast, usually in one or two iterations. This is consistent with the analysis in [28]; see also [22, §15.9].

Next we consider the computation of the extended Krylov subspace. Since $A_1$, as defined in (2.7), is a dense matrix, it is not desirable to form it explicitly. For one part of the extended space, we use the sparse matrix $A$, since $\mathcal{K}(A_1, B_1) = \mathcal{K}(A, B_1)$.
Let $r = \dim \mathcal{N}(A)$. For the other part we separate the deflation into a matrix with $r$ rows and columns, and the difference, as follows. For simplicity, let us describe first the case $r = 1$, and recall that for this symmetric case $A_1 = A - vv^T$, where $v$ spans $\mathcal{N}(A)$. We write $v = u + y$, where $u = v_n e_n$, i.e., a vector with all zeros except the last component, which has the same entry as $v$. Then we write

$$A_1 = A - vv^T = A - (u + y)(u + y)^T = A - uu^T - uy^T - yu^T - yy^T =: A_2 - yy^T.$$ 

It can be appreciated that $A_2$ has the same zero-nozero structure as $A_1$, except for the last row and column, and $A_2$ is sparse and nonsingular. Thus, the LU factorization of $A_2$ should have a very similar fill-in as that of $A$, and one can compute the effect of $A_1$ using the Sherman-Morrison-Woodbury formula. The case of $r > 1$ is similar. Using Matlab notation, let

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ V_2 \end{bmatrix} + \begin{bmatrix} V_1 \\ 0 \end{bmatrix} = U + Y.$$ 

As for the case $r = 1$, we have

$$A_1 = A - VV^T = A - (U + Y)(U + Y)^T = A - UU^T - UY^T - YU^T - YY^T.$$ 

Let $A_2 = A - UU^T - UY^T - YU^T$. Moreover, $UU^T + UY^T + YU^T$ has the following bordered form

$$\begin{bmatrix} 0 & V_1 V_2^T \\ V_2 V_1^T & V_1 V_2^T \end{bmatrix},$$

where $V_1 \in \mathbb{R}^{(n-r) \times r}$, $V_2 \in \mathbb{R}^{r \times r}$. The sparsity structure of $A$ is maintained in $A_2$, so that for small $r$, the fill-in in the LU decomposition of $A_2$ is similar to that of (the singular) $A$. Since $A_2$ is invertible, we can use Sherman-Morrison-Woodbury to compute products of the form $(A - VV^T)^{-1} Z = (A_2 - YY^T)^{-1} Z$.

5. Numerical experiments. We consider several classes of problems, and for each experiment, we vary the order of the matrices. In all cases, we consider data $BB^T$ of rank 7, mostly to illustrate that our proposed method is not restricted to data terms of rank 1.

Example 5.1. The matrix $A$ corresponds to a discretization of the 2D or 3D Laplacian operators with Neumann boundary conditions. Thus, this is a case where the null space of $A$ is unidimensional. The matrix $B$ is of rank 7 and produced randomly.

We set the tolerance for $X_1$ to $10^{-10}$ and that for $A^+ B_1$ to $10^{-9}$. In our numerical tests, we first solve for $X_1$. When the projection method with the extended subspace $\mathcal{K}_{m}^{ext}(A_1, B_1)$ has converged, we use the orthonormal basis of this subspace to obtain the approximation to $A^+ B_1$. Although the accuracy of $A^+ B_1$ may be a little lower than that of $X_1$, in our experience, it suffices for the overall accuracy of the solution. We use this setting in all other numerical experiments. In Tables 5.1 and 5.1 below (as well as in those of the other examples), we report the number of iterations ($k$, which determines the dimension of the projection space), the rank of $X_1^{(k)} = ZZ^T$ (and thus of $X^{(k)}$ here), that of the other term of the solution, and the computational time (in seconds) to compute the null space of $A$, and that for the computation of the solution. Our experiments were performed on an Intel(R) Core(TM) i7-4712HQ machine with CPU at 2.3GHZ and 16GB of RAM. The version of Matlab is 2015a.
Example 5.2. We consider a nonsymmetric matrix $A$ with unidimensional null space. The matrix $A$ corresponds to a discretization of the two-dimensional PDE operator

$$-\Delta u + \vec{a} \nabla u$$

(5.1)

on the unit square with Neumann boundary conditions, using finite elements with triangular elements and quadratic polynomial basis functions. Here $\vec{a} = (1, 2)$, and the matrix $B$ also is produced randomly of rank 7. We use the same tolerance setting of $10^{-9}$ for $Z$ and for $A^+ B_1$. Results of our numerical experiment are given in Table 5.3.

We observe that in this nonsymmetric case, the number of iterations is slightly higher than for the symmetric case in Example 5.1, but that the rank of the solution does not increase much.

Example 5.3. We consider the semi-stability of second order systems, namely systems of the following form

$$M \ddot{q} + (C + G) \dot{q} + K q = 0,$$

(5.2)
where $M, C, G$ and $K$ are $n \times n$ matrices and represent mass, damping, gyroscopic coupling and stiffness parameters, respectively. They are of fundamental importance in the study of vibrational phenomena. In most applications, $M$ is positive definite, $C$ is nonnegative definite, $G$ is skew symmetric and $K$ is symmetric.

The second order system (5.2) can be rewritten as a first order system of the form

$$\dot{x}(t) = Ax(t),$$

(5.3)

where $x(t) = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}$, $A = \begin{bmatrix} 0 & -M^{-1}K \\ -M^{-1}(C + G) & I \end{bmatrix}$.

It can be shown that $\mathcal{N}(A) = \mathcal{N}\left( \begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix} \right)$ and the following theorem holds [3].

**Theorem 5.1.** Let $K$ be positive semidefinite. Then, $A$ as in (5.3) is Lyapunov stable if and only if

$$\text{rank} \left( \begin{bmatrix} K & G \\ 0 & K \\ 0 & C \end{bmatrix} \right) = n + \text{rank}(K).$$

Here, Lyapunov stable means every eigenvalue of $A$ lies in the closed left half complex plane and every eigenvalue of $A$ with zero real part is semi-simple.

In our numerical tests, we take $M$ to be the identity matrix, we set $K = C$ whose first $n - r$ columns are the same as those of the identity matrix and the last $r$ columns are zero vectors. We set $G$ to be the Poisson operator with Dirichlet boundary conditions (which is not skew-symmetric matrix). We chose this setting, so that $\dim(\mathcal{N}(A)) = r$, and $A$ is a nonsymmetric Lyapunov stable matrix with some complex eigenvalues. Thus, if we can prove that $A$ does not have imaginary eigenvalues, then $A$ will be semi-stable. We do so in the following.

**Theorem 5.2.** Let $M = I$ and let $K = C$ whose first $n - r$ columns are those of the identity, and the last $r$ columns are zero. If $G$ is a positive definite matrix, then $A$ in (5.3) is semi-stable.

**Proof.** Let $\begin{bmatrix} u \\ v \end{bmatrix}$ be an eigenvector of $A$, i.e., such that

$$A \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -(K + G) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}.$$ 

Thus, we have $v = \lambda u$ and $-Ku - (K + G)v = \lambda v$. It follows that

$$Ku + \lambda(K + G)u + \lambda^2 u = 0.$$ 

(5.4)

For any unit vector $u$, multiplying by $u^*$ on the left hand side of (5.4), and letting $k = u^* Ku \geq 0$, $g = u^* Gu > 0$, one obtains

$$\lambda^2 + \lambda(k + g) + k = 0.$$ 

Therefore, $\lambda = \frac{-(k + g) \pm \sqrt{(k + g)^2 - 4k}}{2}$. First we note that from the way we construct $K$, $k = n - r$, and since $g > 0$ one has that $(k + g)^2 - 4k < 0$, and thus $A$ can only have complex eigenvalues with negative real part.

Moreover, since $A$ is Lyapunov stable, this implies that the zero eigenvalues are semi-simple. Thus $A$ is semi-stable. $\Box$
Solution of semi-stable Lyapunov equations

Table 5.4
Second order systems, rank $B_1 = 7$, $\dim(N(A)) = 2, 4, 8, 16$

<table>
<thead>
<tr>
<th>$\dim(N(A))$</th>
<th>order of $A$</th>
<th>iter.</th>
<th>$\text{rank}(Z)$</th>
<th>time for $\mathcal{N}(A^T)$</th>
<th>time for $Z$ and $A^+B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>20000</td>
<td>24</td>
<td>183</td>
<td>0.03</td>
<td>6.3</td>
</tr>
<tr>
<td></td>
<td>80000</td>
<td>24</td>
<td>198</td>
<td>0.10</td>
<td>21.6</td>
</tr>
<tr>
<td></td>
<td>180000</td>
<td>24</td>
<td>219</td>
<td>0.26</td>
<td>51.1</td>
</tr>
<tr>
<td>4</td>
<td>20000</td>
<td>24</td>
<td>182</td>
<td>0.03</td>
<td>6.8</td>
</tr>
<tr>
<td></td>
<td>80000</td>
<td>24</td>
<td>200</td>
<td>0.11</td>
<td>23.1</td>
</tr>
<tr>
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<td>24</td>
<td>217</td>
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</tr>
<tr>
<td>8</td>
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<td>0.04</td>
<td>7.49</td>
</tr>
<tr>
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<td>198</td>
<td>0.13</td>
<td>26.4</td>
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<td>0.33</td>
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<tr>
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<td>180000</td>
<td>24</td>
<td>215</td>
<td>0.55</td>
<td>71.2</td>
</tr>
</tbody>
</table>

As in the previous experiments, $B$ is produced randomly with $\text{rank}(B) = 7$, and the tolerance is $10^{-10}$. In Table 5.4, we show results for $A$ with $\dim(N(A)) = 2, 4, 8, 16$.

In these numerical tests, we need to use $\sigma = 10^{-10}$ as a shift to compute the null space. Observe that for this example, increasing the dimension of the null space does not increase the computing time by much.

Example 5.4. Our last example is for equations of the form (3.1), i.e., for a nonsymmetric data term. We use the same matrix $A$ as the Example 5.1, but we produce matrix $B$ and $C$ randomly and separately such that $\text{rank}(B) = \text{rank}(C) = 7$. In other words, $B \neq C$.

Table 5.5
Nonsymmetric data, rank $B_1 = 7$

<table>
<thead>
<tr>
<th>order of $A$</th>
<th>iter.</th>
<th>$\text{rank}(Z_1)$, $\text{rank}(Z_2)$</th>
<th>time for $\mathcal{N}(A)$</th>
<th>time for $Z$ and $A^+B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10000</td>
<td>21</td>
<td>156, 289</td>
<td>0.3</td>
<td>6.2</td>
</tr>
<tr>
<td>15625</td>
<td>23</td>
<td>158, 315</td>
<td>0.6</td>
<td>10.0</td>
</tr>
<tr>
<td>22500</td>
<td>25</td>
<td>164, 340</td>
<td>1.1</td>
<td>17.0</td>
</tr>
<tr>
<td>30625</td>
<td>27</td>
<td>170, 364</td>
<td>1.7</td>
<td>26.4</td>
</tr>
<tr>
<td>40000</td>
<td>28</td>
<td>171, 376</td>
<td>3.0</td>
<td>40.2</td>
</tr>
<tr>
<td>62500</td>
<td>31</td>
<td>171, 414</td>
<td>7.8</td>
<td>71.0</td>
</tr>
<tr>
<td>90000</td>
<td>34</td>
<td>182, 448</td>
<td>15.3</td>
<td>121.5</td>
</tr>
<tr>
<td>122500</td>
<td>36</td>
<td>182, 474</td>
<td>26.7</td>
<td>193.9</td>
</tr>
<tr>
<td>160000</td>
<td>38</td>
<td>192, 495</td>
<td>39.9</td>
<td>309.4</td>
</tr>
</tbody>
</table>

It can be observed in Table 5.5 that both, the rank of the solution, as well as the computational time are about double than those of Example 5.1. This is of course to be expected, since we project onto the sum of two extended subspaces.

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REFERENCES


