On Necessary Conditions for Convergence of Stationary Iterative Methods for Hermitian Semidefinite Linear Systems

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ON NECESSARY CONDITIONS FOR CONVERGENCE OF STATIONARY ITERATIVE METHODS FOR HERMITIAN SEMIDEFINITE LINEAR SYSTEMS*

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Abstract. In an earlier paper [*SIAM J. Matrix Anal. Appl.* vol. 30 (2008), 925–938] we gave sufficient conditions in terms of an energy seminorm for the convergence of stationary iterations for solving linear systems whose coefficient matrix is Hermitian and positive semidefinite. In this paper we show in which cases these conditions are also necessary, and show that they are not necessary in others.

Key words. linear systems, singular systems, stationary iterative methods, seminorm, convergence analysis

AMS subject classifications. 65F10, 65F20.

1. Introduction.

We consider the (singular) linear system

$$Ax = b, \tag{1.1}$$

where the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is assumed to be Hermitian and positive semidefinite.[1]

If A is large and sparse, iterative methods for solving (1.1) are the standard approach. In this paper, we focus on stationary iterative methods, including, for example, certain algebraic multigrid methods and additive and multiplicative Schwarz methods. Sometimes, these iterations are accelerated by using them as preconditioners to Krylov subspace methods like Conjugate Gradients. While we do not consider the latter aspect in any detail in this work, let us just mention that one usually assumes convergence of the preconditioner as a prerequisite in this context, so our work is relevant in this case as well.

For a stationary iterative method, one usually considers a splitting $A = M - (M - A)$ with M nonsingular, a resulting iteration matrix $H = I - M^{-1}A$, together with the iteration

$$x^{k+1} = Hx^k + M^{-1}b. \tag{1.2}$$

As is well known, the iteration (1.2) converges to the unique solution of (1.1) for any initial vector x^0 if and only if $\rho(H) < 1$, where $\rho(H)$ is the spectral radius of H, and such matrix H is termed *convergent*[2]; see, e.g., [3], [19]. We mention in passing that when A is nonsingular it holds that for a given matrix A and a given convergent matrix H, there exists a unique corresponding nonsingular matrix M such that $H = I - M^{-1}A$ [10].

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[1] The case of A real is treated exactly in the same manner. We discuss the case of A complex for full generality, though the proofs are essentially the same in both cases.

[2] We note that in some papers such matrix is called *zero-convergent* since $\lim_{k\to\infty} H^k = O$.

A *sufficient* condition for the convergence of the iteration (1.2) when A is Hermitian and positive definite is given by the following result in [20, Satz 1, p. 156] (see also [7, p. 111], [8, p. 21]), now commonly called the *P-regular splitting theorem.* A splitting of (the Hermitian positive definite matrix) $A = M - (M - A)$ is called *P-regular* if $M + M^H - A$ is positive definite [14].

THEOREM 1.1. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite, and $A = M - (M - A)$ be a P-regular splitting. Then $\rho(H) < 1$, with $H = I - M^{-1}A$.*

The classical proof uses Stein's theorem (see, e.g., [14, 7.1.8]). Below we offer essentially the same proof but highlight the fact that one not only has $\rho(H) < 1$ but even $\|H\|_A < 1$ with $\|H\|_A$ the operator norm induced by the energy norm

$$\|x\|_A = \langle x, x\rangle_A^{1/2}, \quad \text{where} \quad \langle x, x\rangle_A = \langle Ax, x\rangle \quad (= \langle x, Ax\rangle), \tag{1.3}$$

$\langle x, y\rangle$ denoting the standard Euclidian inner product. This result is worth mentioning because it implies that the errors of the iterates from (1.2) then decrease monotonically when measured in the energy norm which is a canonical norm for the system (1.1). Moreover, the "energy norm version" of Theorem 1.1 also allows for a canonical converse, namely that $\|H\|_A < 1$ implies that the splitting is P-regular.

THEOREM 1.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Then, $A = M - (M - A)$ is a P-regular splitting if and only if $\|H\|_A < 1$.*

Proof. We write

$$H^H AH = (I - M^{-1}A)^H A(I - M^{-1}A) = A - AM^{-H}(M + M^H - A)M^{-1}A.$$

Then, $\|Hu\|_A^2 = u^H H^H AHu < u^H Au = \|u\|_A^2$ for all vectors $u \neq 0$ if and only if $M + M^H - A$ is positive definite. □

We mention that other converses of Theorem 1.1 are possible. A typical result is that for a given P-regular splitting and A Hermitian, $\rho(H) < 1$ implies that A is positive definite, see, e.g., [9], [14, E71.9], and further results of this kind when A is non-Hermitian, see, e.g., [1], [7, p. 111], and references therein.

Several authors have given sufficient conditions for convergence in the more general setting of A being positive semidefinite; and these are reviewed in section 2, where we emphasize on conditions based on the A-seminorm, defined below. In section 3 we then answer the following question: In which cases are these sufficient conditions also necessary?

2. The semidefinite case and the A-seminorm. When A is semidefinite, the expression (1.3) defines a seminorm, and not a norm. In this section we investigate in detail the role of the induced *operator A-seminorm* in sufficient conditions for the convergence of iterations based on splittings. Except for Proposition 2.5 the results of this section are not new; the systematic use of the operator seminorm in the formulation of the results, however, provides a unifying approach we believe to have an interest of its own. We demosntrate this when discussing various convergence results from the literature in the light of Theorem 2.4 below.

Denoting by $\mathrm{Null}(A)$ the nullspace of A and by $\mathrm{Range}(A) = \mathrm{Null}(A)^\perp$ its range, we assume that $b \in \mathrm{Range}(A)$. This implies that the solution set of (1.1) is nonempty and it is given as an affine space $x^* + \mathrm{Null}(A)$ for some $x^* \in \mathbb{C}^n$ solution of (1.1).

Following [5], we consider the general situation in which the iteration matrix H for (1.1) is of the form

$$H = I - \widetilde{M}A, \tag{2.1}$$

where $\widetilde{M} \in \mathbb{C}^{n\times n}$ is a matrix which might be singular. The matrices H and $\widetilde{M}$ induce the iteration

$$x^{k+1} = Hx^k + \widetilde{M}b. \tag{2.2}$$

We remark that, as opposed to the non-singular case, for given A and H, the matrix $\widetilde{M}$ in (2.1) is not necessarily unique; cf. [2]. A minimal assumption on $\widetilde{M}$ is that it be injective on $\mathrm{Range}(A)$, because otherwise, if $b \in \mathrm{Range}(A), b \neq 0$ is in $\mathrm{Null}(\widetilde{M})$ and $x^0 = 0$, the iteration (2.2) produces iterates which are all equal to 0, i.e. the iteration does not converge to a solution of (1.1).

The general form of the iteration operator from (2.1) applies in particular to iterations induced by splittings of the form $A = M - (M - A)$, M nonsingular, in which case $\widetilde{M}$ is taken to be M^{-1}. There are iterations which can be interpreted as being of the form (2.1) with $\widetilde{M} = M^\dagger$, the Moore-Penrose pseudoinverse of some singular matrix M; see [4], [11], [12], where such iterations are studied. This situation occurs in particular in the analysis of Schwarz iterations where the artificial boundary conditions between subdomains are of Neumann or Robin type; see, e.g., [13], [16], [18].

Convergence of the iteration (2.2) is equivalent to H being semiconvergent according to the following definition[3]; see, e.g., [3], [4], [17].

Definition 2.1. *A matrix $H \in \mathbb{C}^{n\times n}$ is called semiconvergent, if $\rho(H) = 1$, $\lambda = 1$ is the only eigenvalue of modulus 1 and $\lambda = 1$ is a semisimple eigenvalue of H, i.e., its geometric multiplicity is equal to its algebraic multiplicity.*

The A-seminorm on $\mathbb{C}^n$ induces an operator A-seminorm $|||H|||_A$ on $\mathbb{C}^{n\times n}$ via

$$|||H|||_A = \max_{x\in \mathrm{Range}(A)} \frac{\|Hx\|_A}{\|x\|_A} \quad \Big(= \max_{x\in \mathrm{Range}(A), \|x\|_2=1} \frac{\|Hx\|_A}{\|x\|_A} \Big). \tag{2.3}$$

In the positive definite case, $\|H\|_A < 1$ implies $\rho(H) < 1$ and thus convergence of the iteration (2.2). In the semidefinite case, the analogous result holds with the operator A-seminorm.

Theorem 2.2. *Let A be Hermitian and positive semidefinite. Let $H = I - \widetilde{M}A$ be the iteration operator of the iteration (2.2), and let $|||H|||_A < 1$. Then*

(i) $\widetilde{M}$ is injective on $\mathrm{Range}(A)$.

(ii) H is semiconvergent.

Proof. This result was already given in [5] with the assumption $|||H|||_A < 1$ replaced by

$$x \notin \mathrm{Null}(A) \Longrightarrow \|Hx\|_A < \|x\|_A. \tag{2.4}$$

As was remarked in [15], the two are equivalent, though: With Π denoting the orthogonal projection onto $\mathrm{Range}(A)$ we have $\|x\|_A = \|\Pi x\|_A$ for all x. Since $Hy = y$ for $y \in \mathrm{Null}(A)$, we also have $Hx = H(I - \Pi)x + H\Pi x = (I - \Pi)x + H\Pi x$ for all x. This implies that for any x we have $\|x\|_A = \|\Pi x\|_A$ as well as $\|Hx\|_A = \|H\Pi x\|_A$, and therefore (2.4) is equivalent to

$$x \in \mathrm{Range}(A) \Longrightarrow \|Hx\|_A < \|x\|_A. \tag{2.5}$$

Since in the definition of $|||\cdot|||_A$ from (2.3) we can restrict x to the intersection of $\mathrm{Range}(A)$ with the unit sphere, a compact set, (2.5) is equivalent to $|||H|||_A < 1$. □

[3] In some papers such a matrix is simply called *convergent*.

We proceed by stating the counterparts to Theorems 1.1 and 1.2 in the semidefinite case. We say that a Hermitian matrix $B \in \mathbb{C}^{n\times n}$ is positive definite *on a subspace* V of $\mathbb{C}^n$ if $\langle Bx, x\rangle > 0$ for all $x \in V, x \neq 0$.

THEOREM 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let M be nonsingular, let $H = I - M^{-1}A$ and assume that $M + M^H - A$ is positive definite on* $\mathrm{Range}(M^{-1}A)$. *Then H is semiconvergent.*

This result can be found in [20, Hauptsatz, p. 160].

In analogy to the positive definite case, the assumptions of Theorem 2.3 already imply that $|||H|||_A < 1$, and a corresponding converse result also holds. Moreover, we can deal with an iteration matrix H of the general form (2.1), i.e. we can replace M^{-1} by a possibly singular matrix $\widetilde{M}$ which is injective on $\mathrm{Range}(A)$. Note that $\widetilde{M}$ is injective on $\mathrm{Range}(A)$ if $\mathrm{Null}(\widetilde{M}A) = \mathrm{Null}(A)$ or, equivalently, if there exists a matrix M such that $M\widetilde{M}A = A$. The precise result is as follows and was given in [5].

THEOREM 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and positive semidefinite. Let $\widetilde{M} \in \mathbb{C}^{n\times n}$ be injective on* $\mathrm{Range}(A)$*, let $M \in \mathbb{C}^{n\times n}$ satisfy*

$$M\widetilde{M}A = A \tag{2.6}$$

and let $H = I - \widetilde{M}A$. Then $|||H|||_A < 1$ if and only if $M + M^H - A$ is positive definite on $\mathrm{Range}(\widetilde{M}A)$.

The following proposition clarifies that the various possible choices for M in (2.6) have no impact on the assertion of the theorem.

PROPOSITION 2.5. *All matrices M with $M\widetilde{M}A = A$ induce an identical quadratic form $\langle (M + M^H - A)x, x\rangle$ on* $\mathrm{Range}(\widetilde{M}A)$.

Proof. Let $M, \underline{M} \in \mathbb{C}^{n\times n}$ be two matrices which differ only by their action on a space complementary to $\mathrm{Range}(\widetilde{M}A)$. Then $\langle Mx, x\rangle = \langle \underline{M}x, x\rangle$ for $x \in \mathrm{Range}(\widetilde{M}A)$, and, similarly, $\langle M^H x, x\rangle = \langle x, Mx\rangle = \langle x, \underline{M}x\rangle = \langle \underline{M}^H x, x\rangle$ for $x \in \mathrm{Range}(\widetilde{M}A)$. $\square$

Proposition 2.5 says in particular that if $\widetilde{M}$ is injective on $\mathrm{Range}(A)$, then either for all M with $M\widetilde{M}A = A$ the matrix $M + M^H - A$ is positive definite on $\mathrm{Range}(\widetilde{M}A)$ or $M + M^H - A$ is positive definite for no such M.

We end this section with a discussion on two results known from the literature which appear as special cases of Theorem 2.4. We start with [4, Theorem 3.4], where it is shown that for the splitting $A = M - (M - A)$, and $\widetilde{M}$ the Moore-Penrose inverse $M^\dagger$, if $\mathrm{Range}(A) \subseteq \mathrm{Range}(M)$ and $M + M^H - A$ is positive definite on $\mathrm{Range}(M^\dagger A)$, then $|||H|||_A < 1$. In this manner, this result in [4] also clarifies a possible misunderstanding in [11, Theorem 4.4]. Since $\mathrm{Range}(A) \subseteq \mathrm{Range}(M)$ implies that $M^\dagger$ is injective on $\mathrm{Range}(A)$ [4, Theorem 3.3], we see that this result is a special case of Theorem 2.4 with $\widetilde{M} = M^\dagger$.

The paper [12] allows $\widetilde{M}$ to be a general reflexive inverse of M, where $A = M - N$, i.e., a matrix for which $M\widetilde{M}M = M$ and $\widetilde{M}M\widetilde{M} = \widetilde{M}$. The paper considers the iteration

$$x^{k+1} = \widetilde{M}Nx^k + \widetilde{M}b, \tag{2.7}$$

which conincides with (2.2) if $x^0 \in \mathrm{Range}(\widetilde{M})$. This follows since $\widetilde{M}N\widetilde{M} = \widetilde{M} - \widetilde{M}A\widetilde{M} = (I - \widetilde{M}A)\widetilde{M}$, and the fact that $x^k \in \mathrm{Range}(\widetilde{M})$ for all iterates x^k from (2.7). If $x^0 \notin \mathrm{Range}(\widetilde{M})$, all subsequent iterates of (2.7) are from $\mathrm{Range}(\widetilde{M})$, so that x^k from (2.7) is identical to the $(k-1)$st iterate from (2.2), if (2.2) takes as its initial vector the first iterate of (2.7). Consequently, convergence of (2.2) implies convergence of (2.7). Theorem 3.2 in [12] now shows that if $\operatorname{Range}(A) \subseteq \operatorname{Range}(M)$, $M+M^H-A$ is symmetric positive definite on $\operatorname{Range}(I-\widetilde{M}N)$ and $\operatorname{Range}(\widetilde{M}A) \subseteq \operatorname{Range}(I-\widetilde{M}N)$, the iteration (2.7) converges for any starting vector x^0 and $b \in \operatorname{Range}(A)$. Here, as before, $\operatorname{Range}(A) \subseteq \operatorname{Range}(M)$ implies that $\widetilde{M}$ is injective on $\operatorname{Range}(A)$, and if $M+M^H-A$ is symmetric positive definite on $\operatorname{Range}(I-\widetilde{M}N)$ with $\operatorname{Range}(\widetilde{M}A) \subseteq \operatorname{Range}(I-\widetilde{M}N)$, it is also symmetric positive definite on $\operatorname{Range}(\widetilde{M}A)$. So Theorem 2.4 applies and we have again $|||I-\widetilde{M}A|||_A < 1$.

3. Sufficient conditions. In this section we present a new result which shows that the sufficient conditions presented in Theorem 2.4 are also necessary when $\widetilde{M}$ is Hermitian. We also show that when $\widetilde{M}$ is not Hermitian, this is not necessarily the case, except for $n=2$.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive semidefinite. Let $H = I-\widetilde{M}A$ be such that H is semiconvergent. Assume that $\widetilde{M}$ is injective on* $\operatorname{Range}(A)$ *and let M be such that $M\widetilde{M}A = A$.*

(i) The hypotheses do not necessarily imply that $M+M^H-A$ is positive definite on $\operatorname{Range}(\widetilde{M}A)$.

(ii) Let $n=2$. Then $M+M^H-A$ is positive definite on $\operatorname{Range}(\widetilde{M}A)$ *(and thus, by Theorem 2.4, $|||H|||_A < 1$).*

(iii) Assume that $\widetilde{M}$ is Hermitian. Then $|||H|||_A < 1$ (and thus, by Theorem 2.4, $M+M^H-A$ is positive definite on $\operatorname{Range}(\widetilde{M}A)$*).*

Proof. (i) We provide an example inspired by one in [1].

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{M} = \begin{bmatrix} 1/2 & 1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows then that $M\widetilde{M}A = A$, and that

$$H = I-\widetilde{M}A = \begin{bmatrix} 1/2 & -1 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is semiconvergent. But we have that

$$M+M^H-A = \begin{bmatrix} 3 & -4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is not positive definite on $\operatorname{Range}(\widetilde{M}A) = \operatorname{Range}(A) = \operatorname{span}\{(1,0,0)^T, (0,1,0)^T\}$. By Proposition 2.5, for any matrix $\underline{M}$ with $\underline{M}\widetilde{M}A = A$ the quadratic forms $\langle (M+M^H-A)x, x\rangle$ and $\langle (\underline{M}+\underline{M}^H-A)x, x\rangle$ coincide on $\operatorname{Range}(\widetilde{M}A)$, implying that for any such $\underline{M}$ the matrix $\underline{M}+\underline{M}^H-A$ is not positive definite on $\operatorname{Range}(\widetilde{M}A)$.
(ii) If $A = 0 \in \mathbb{C}^{2\times 2}$, we have $H = I$ which is semiconvergent, and for any M the matrix $M+M^H-A$ is trivially positive definite on $\operatorname{Range}(\widetilde{M}A) = \{0\}$.

We now turn to the case where the rank of A is 1. By a change of basis, without loss of generality, we can assume that

$$A = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}, \quad c \neq 0.$$

$\widetilde{M}$ being injective on Range$(A) = \text{span}\{(1,0)^T\}$ implies that

$$\widetilde{M} = \begin{bmatrix} a & e \\ b & f \end{bmatrix}$$

for some nonzero a. Thus, we have that

$$H = I - \widetilde{M}A = \begin{bmatrix} 1-ac & 0 \\ -bc & 1 \end{bmatrix}.$$

Since H is semiconvergent, this implies that $0 < ac < 2$. Condition (2.6) gives

$$M = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix},$$

if $b \neq 0$ (and zero in the (2,2) position otherwise) as a possible choice for M. Then

$$M + M^H - A = \begin{bmatrix} 2/a - c & 0 \\ 0 & 2/b \end{bmatrix} \quad \left(\text{or } \begin{bmatrix} 2/a - c & 0 \\ 0 & 0 \end{bmatrix}\right),$$

which, since $2/a - c > 0$, is positive definite on Range$(\widetilde{M}A)$ = Range(A) = span$\{(1,0)^T\}$. By Proposition 2.5, $M + M^H - A$ is also positive definite on Range$(\widetilde{M}A)$ for any other choice of M with $M\widetilde{M}A = A$.

The case when the rank of A is 2 does not apply here, since we assume A to be semidefinite and not definite.

(iii) Let $\hat{H} = I - A^{1/2}\widetilde{M}A^{1/2}$ and note that with $C = \widetilde{M}A^{1/2}$ we have

$$\hat{H} = I - A^{1/2}C, \;\; H = I - CA^{1/2}.$$

For any two square matrices K and L of the same size, the products KL and LK have the same spectrum, see, e.g., [6, Theorem 1.3.20]. Thus, $A^{1/2}C$ and $CA^{1/2}$ share the same spectrum, and so do $\hat{H}$ and H. Since H is semi-convergent, we have

$$\text{spec}(\hat{H}) = \text{spec}(H) \subset \{1\} \cup [-\gamma, \gamma] \text{ with } \gamma \in [0,1). \tag{3.1}$$

We also note that Range(A) = Range$(A^{1/2})$ and Null$(A^{1/2})$ = Null(A).

We now first show that $\widetilde{M}$ is positive definite on Range(A), i.e.

$$\langle \widetilde{M}y, y\rangle > 0 \text{ for } y \in \text{Range}(A), y \neq 0. \tag{3.2}$$

If there were a vector $y \in$ Range(A) such that $\langle \widetilde{M}y, y\rangle < 0$, then $y = Aw = A^{1/2}x$ with $x = A^{1/2}w \in$ Range(A), and thus $\langle \widetilde{M}A^{1/2}x, A^{1/2}x\rangle < 0$. We can normalize x to $\|x\|_2 = 1$ and thus see that the Hermitian matrix $\hat{H} = I - A^{1/2}\widetilde{M}A^{1/2}$ has a Rayleigh quotient larger than 1. This is impossible, since 1 is the largest eigenvalue of $\hat{H}$, see (3.1). Moreover, if $y \in$ Range(A) is a vector such that $\langle \widetilde{M}y, y\rangle = 0$, then $\widetilde{M}^{1/2}y = 0$ and thus $\widetilde{M}y = 0$ which implies $y = 0$ since $\widetilde{M}$ is assumed to be injective on Range(A). We have thus proved (3.2).

Now let $\hat{H}x = x$. Then $A^{1/2}\widetilde{M}A^{1/2}x = 0$ and thus $\langle \widetilde{M}A^{1/2}x, A^{1/2}x\rangle = 0$. Since $\widetilde{M}$ is positive definite on Range(A) this implies $A^{1/2}x = 0$, i.e., $x \in$ Null(A). Since, trivially, we also have $\hat{H}x = x$ whenever $x \in$ Null(A), we get

$$\hat{H}x = x \Longleftrightarrow x \in \text{Null}(A). \tag{3.3}$$

To finish the proof we note that since $A^{1/2}Hx = \hat{H}A^{1/2}x$ for all x, we have

$$
\begin{aligned}
|||H|||_A &= \max_{x\in\mathrm{Range}(A)} \frac{\|Hx\|_A}{\|x\|_A} = \max_{x\in\mathrm{Range}(A)} \frac{\|A^{1/2}Hx\|}{\|A^{1/2}x\|} \\
&= \max_{x\in\mathrm{Range}(A)} \frac{\|\hat{H}A^{1/2}x\|}{\|A^{1/2}x\|} = \max_{y\in\mathrm{Range}(A)} \frac{\|\hat{H}y\|}{\|y\|}\,. \qquad (3.4)
\end{aligned}
$$

Since $\hat{H}$ is Hermitian, it admits an orthonormal basis of eigenvectors. By (3.3), $\mathrm{Range}(A) = \mathrm{Null}(A)^{\perp}$ is spanned by the eigenvectors of $\hat{H}$ which belong to eigenvalues $\lambda \neq 1$ which, by (3.1) are those from $[-\gamma, \gamma]$. In (3.4) we thus have

$$
\max_{y\in\mathrm{Range}(A)} \frac{\|\hat{H}y\|}{\|y\|} = \gamma \ < 1. \qquad \square
$$

4. Conclusions. In earlier works, it was shown that for a linear system with a Hermitian positive semidefinite coefficient matrix A, a sufficient condition for the convergence of a stationary iteration with iteration matrix $I - \widetilde{M}A$, with $\widetilde{M}$ injective on $\mathrm{Range}(A)$, is that the splitting $A = M - (M - A)$ be a P-regular splitting for a matrix M with $M\widetilde{M}A = A$. In this paper, we have shown that with $\widetilde{M}$ Hermitian, this sufficient condition is also necessary and that convergence of the iteration matrix is then equivalent to its A-seminorm being less than 1. We have also shown that when $\widetilde{M}$ is not Hermitian, P-regularity of the splitting is not always a necessary condition for convergence.

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