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Abed Elhashash and Daniel B. Szyld

Report 07-08-17
August 2007

This is a slightly revised version of the original report dated 17 August 2007.

This report is available in the World Wide Web at
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GENERALIZATIONS OF M-MATRICES WHICH MAY NOT HAVE A NONNEGATIVE INVERSE*

ABED ELHASHASH† AND DANIEL B. SZYLD†

Dedicated to Richard S. Varga on the occasion of his 80th birthday

Abstract. Generalizations of M-matrices are studied, including the new class of GM-matrices. The matrices studied are of the form $sI - B$ with B having some Perron-Frobenius property, but not necessarily being nonnegative. Results for these classes of matrices are shown, which are analogous to those known for M-matrices. Also, various splittings of a GM-matrix are studied along with conditions for their convergence.

Key words. Generalized M-matrices, Perron-Frobenius property.

AMS subject classifications. 15A48

1. Introduction.

Nonnegative matrices possess the Perron-Frobenius property, i.e., each nonnegative matrix has a dominant positive eigenvalue that corresponds to a nonnegative eigenvector. In a recent paper [6], we studied matrices that are not necessarily nonnegative yet possess the Perron-Frobenius property. Closely related to this subject is the subject of M-matrices. A matrix $A \in \mathbb{R}^{n\times n}$ is called an M-matrix if it can be expressed as $A = sI - B$ where B is nonnegative and has a spectral radius $\rho(B) \le s$.

In this paper, we study generalizations of M-matrices of the form $A = sI - B$ where B and B^T possess the Perron-Frobenius property and $\rho(B) \le s$. We call such matrices GM-matrices. We also study other generalizations of this type and present some of their properties which are counterparts to those of M-matrices.

Recall that a matrix B is said to be *eventually nonnegative* (*eventually positive*) if $B^k \ge 0$ ($B^k > 0$, respectively) for all $k \ge p$ for some positive integer p.

Among the generalizations of M-matrices we study are matrices of the form $A = sI-B$ with $\rho(B) \le s$ and B being an eventually nonnegative or an eventually positive matrix. Johnson and Tarazaga [15] termed the latter class , pseudo-M-matrices. Le and McDonald [16] studied the case where B is an irreducible eventually nonnegative matrix. We mention also other generalizations of M-matrices not considered in this paper; namely, where B leaves a cone invariant (see, e.g., [22], [24]) or for rectangular matrices; see, e.g., [20].

It is well-known that the inverse of a nonsingular M-matrix is nonnegative [1], [23] (and we prove an analogous result for GM-matrices in section 3). This property leads to the natural question: for which nonnegative matrices is the inverse an M-matrix? This question and related topics were extensively studied; see, e.g., [4], [5], [7]–[10], [12]–[14], [17]. In section 3, we study analogous questions, such as: for which matrices having the Perron-Frobenius property is the inverse a GM-matrix?

Another aspect we address (in section 4) is the study of splittings $A = M - N$ of a GM-matrix A and of conditions for their convergence.

*this version is dated 27 August 2007.

†Department of Mathematics, Temple University (038-16), 1805 N. Broad Street, Philadelphia, Pennsylvania 19122-6094, USA (abed@temple.edu, szyld@temple.edu). Supported in part by the U.S. National Science Foundation under grant CCF-0514889, and by the U.S. Department of Energy under grant DE-FG02-05ER25672.

2. Notation and Preliminaries. We say that a real or complex matrix A is nonnegative (positive, nonpositive, negative, respectively) if it is entry-wise nonnegative (positive, nonpositive, negative, respectively) and we write $A \geq 0$ ($A > 0$, $A \leq 0$, $A < 0$, respectively). This notation and nomenclature is also used for vectors. If v is a nonzero and nonnegative column or row vector then we say v is *semipositive*. The spectral radius of a matrix A is denoted by $\rho(A)$. The spectrum of a matrix A is denoted by $\sigma(A)$. We call an eigenvalue of A a simple eigenvalue if its algebraic multiplicity in the characteristic polynomial is 1. We call an eigenvalue $\lambda \in \sigma(A)$ *dominant* if $|\lambda| = \rho(A)$. We call an eigenvalue $\lambda \in \sigma(A)$ *strictly dominant* if $|\lambda| > |\mu|$ for all $\mu \in \sigma(A)$, $\mu \neq \lambda$. The ordinary eigenspace of A for the eigenvalue λ is denoted by $E_\lambda(A)$. By definition, $E_\lambda(A) = \mathcal{N}(A - \lambda I)$, the null space of $A - \lambda I$. The nonzero vectors in $E_\lambda(A)$ are called ordinary eigenvectors of A corresponding to λ.

We say that a matrix $A \in \mathbb{R}^{n\times n}$ has the *Perron-Frobenius property* if the spectral radius is a positive eigenvalue that has an entry-wise nonnegative eigenvector. Also, we say that a matrix $A \in \mathbb{R}^{n\times n}$ has the *strong Perron-Frobenius property* if the spectral radius $\rho(A)$ is a simple positive eigenvalue that is strictly larger in modulus than any other eigenvalue and there is an entry-wise positive eigenvector corresponding to $\rho(A)$. By PF_n we denote the collection of matrices that are eventually positive. It turns out that $A \in PF_n$ if and only if A and A^T possess the strong Perron-Frobenius property; see, e.g., [15] and [18]. By WPF_n we denote the collection of matrices A for which both A and A^T possess the Perron-Frobenius property. The containments in the following statement are proper; see [6, Section 5].

$$PF_n \subset \{\text{nonnilpotent eventually nonnegative matrices}\} \subset WPF_n.$$

We say that $A \in \mathbb{C}^{1\times 1}$ is reducible if $A = [0]$. We say that $A \in \mathbb{C}^{n\times n}$ ($n \geq 2$) is reducible if A is permutationally similar to $\begin{bmatrix} B & O \\ C & D \end{bmatrix}$ where B and D are square matrices. We say that a matrix $A \in \mathbb{C}^{n\times n}$ ($n \geq 1$) is irreducible if A is not reducible.

For an $n \times n$ matrix A, we define the graph $G(A)$ to be the graph with vertices $1, 2, \ldots, n$ in which there is an edge (i, j) if and only if $a_{ij} \neq 0$. We say vertex i has access to vertex j if $i = j$ or else if there is a sequence of vertices $(v_1, v_2, \cdots, v_r)$ such that $v_1 = i$, $v_r = j$ and (v_i, v_{i+1}) is an edge in $G(A)$ for $i = 1, \ldots, r-1$. If i has access to j and j has access to i then we say i and j communicate. Equivalence classes under the communication relation on the set of vertices of $G(A)$ are called the *classes of* A. By $A[\alpha]$ we denote the principal submatrix of $A \in \mathbb{R}^{n\times n}$ indexed by $\alpha \subseteq \{1, 2, \ldots, n\}$. The graph $G(A[\alpha])$ is called a strong component of $G(A)$ whenever α is a class of A. We say that $G(A)$ is strongly connected whenever A has one class, or equivalently whenever A is irreducible. We call a class α basic if $\rho(A[\alpha]) = \rho(A)$. We call a class α initial if no vertex in any other class β has access to any vertex in α.

A matrix $A \in \mathbb{R}^{n\times n}$ is a Z-matrix if A can be expressed in the form $A = sI - B$ where s is a positive scalar and B is a nonnegative matrix. Moreover, if $A = sI - B$ is a Z-matrix such that $\rho(B) \leq s$, then we call A an M-matrix.

If $A \in \mathbb{R}^{n\times n}$ can be expressed as $A = sI - B$ where $B \in WPF_n$, then we call A
- a GZ-matrix.
- a GM-matrix if $0 < \rho(B) \leq s$.
- an EM-matrix if $0 < \rho(B) \leq s$ and B is eventually nonnegative.
- a pseudo-M-matrix if $0 < \rho(B) < s$ and $B \in PF_n$ [15].

When the inverse of a matrix C is a GM-matrix then we call C an inverse GM-matrix.

It follows directly from the definitions that every M-matrix is an EM-matrix, that every EM-matrix is a GM-matrix, and that every pseudo-M-matrix is an EM-matrix. We show by examples below that the converses do not hold.

Furthermore, an M-matrix may not be a pseudo-M-matrix. Consider, for example, a reducible M-matrix. We illustrate the relations among the different sets of matrices in Figure 2.1.

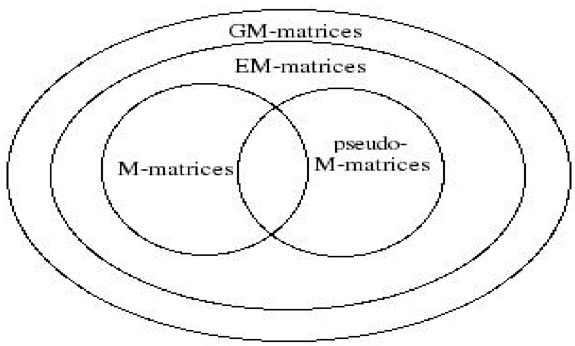

FIG. 2.1. *This diagram summarizes the relations between the sets of various generalizations of M-matrices using the Perron-Frobenius property.*

EXAMPLE 2.1.

$$\text{Let } A = sI - B \text{ where } B = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -1 & 1 & 2 & 2 \end{bmatrix} \text{ and } s > 4.$$

Note that matrix B, which is taken from [3, Example 4.8], is a reducible nonnilpotent eventually nonnegative matrix with $\rho(B) = 4$. Hence, A is an EM-matrix. Since A is reducible, it follows that, for any positive scalar δ, we have $\delta I - A$ reducible and any power of $\delta I - A$ reducible. Hence, for any positive scalar δ, the matrix $\delta I - A$ is not eventually positive (i.e. $(\delta I - A) \notin PF_6$). And thus, A is not a pseudo-M-matrix. Moreover, A is not an M-matrix because A has positive off-diagonal entries.

EXAMPLE 2.2.

$$\text{Let } A = sI - B \text{ where } B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \text{ and } s > 2.$$

Note that $\rho(B) = 2$ is an eigenvalue having $[1\ 1\ 0\ 0]^T$ a right and a left eigenvector. Hence, $B \in WPF_4$ and A is a GM-matrix. However, B is not eventually nonnegative because the lower right 2×2 block of B keeps on alternating signs. Moreover, for any positive scalar δ, the lower 2×2 block of $\delta I - A$ is the matrix $C = \begin{bmatrix} \delta - s - 1 & -1 \\ -1 & \delta - s - 1 \end{bmatrix}$. Note that for any positive integer k, the lower 2×2

block of $(\delta I - A)^k$ is the matrix C^k which is, using an induction argument, the matrix $\frac{1}{2}\begin{bmatrix} (\delta-s-2)^k+(\delta-s)^k & (\delta-s-2)^k-(\delta-s)^k \\ (\delta-s-2)^k-(\delta-s)^k & (\delta-s-2)^k+(\delta-s)^k \end{bmatrix}$. It is easy to see that for any choice of a positive scalar δ the matrix $\delta I - A$ is not eventually nonnegative because the (2,1)-entry of C^k are always negative for odd powers k.

3. Properties of GM-matrices. In this section, we generalize some results known for M-matrices to GM-matrices. For example, if A is a nonsingular M-matrix, then A^{-1} is nonnegative; see, e.g., [1], [23]. We show analogous results for GM- and pseudo-M-matrices. However, we show by an example that no analogous results for EM-matrices hold.

Theorem 3.1. *Let A be a matrix in $\mathbb{R}^{n\times n}$ whose eigenvalues (when counted with multiplicity) are arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Then the following statements are equivalent:*

(i) A is a nonsingular GM-matrix.

(ii) $A^{-1} \in WPF_n$ and $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$.

Proof. Suppose first that $A = sI - B$ is a nonsingular GM-matrix ($B \in WPF_n$ and $0 < \rho(B) < s$). Then, there are semipositive vectors v and w such that $Bv = \rho(B)v$ and $w^T B = \rho(B)w^T$. This implies that $A^{-1}v = (s-\rho(B))^{-1}v$ and that $w^T A^{-1} = (s-\rho(B))^{-1}w^T$. Thus, v and w are eigenvectors of A^{-1} and furthermore $\rho(A^{-1}) = |\lambda_n|^{-1} = (s-\rho(B))^{-1} > 0$. Therefore, $A^{-1} \in WPF_n$. Moreover, $|\lambda_n| = \lambda_n$, i.e., $Re(\lambda_n) > 0$ and $Im(\lambda_n) = 0$, otherwise, if $Re(\lambda_n) \leq 0$ then the eigenvalue $(s-\lambda_n) \in \sigma(B)$ satisfies $|s-\lambda_n| > |s-|\lambda_n|| = \rho(B)$, which is a contradiction. Or, if $Re(\lambda_n) > 0$ but $Im(\lambda_n) \neq 0$, then again, $|s-\lambda_n| > |s-|\lambda_n|| = \rho(B)$, which is a contradiction. Therefore, $|\lambda_n| = \lambda_n > 0$. Similarly, one could show that if $|\lambda_i| = \lambda_n$ for some $i \in \{1,\ldots,n-1\}$ then $\lambda_i = \lambda_n > 0$. Furthermore, suppose that $\lambda_n \geq Re(\lambda_i)$ for some $\lambda_i \neq \lambda_n$, then $|\lambda_i| > \lambda_n$ (otherwise, $\lambda_i = \lambda_n$). If $Re(\lambda_i) = \lambda_n$, then $|\lambda_i| > Re(\lambda_i)$, therefore $|Im(\lambda_i)| > 0$. Thus,

$$|s-\lambda_i| = \sqrt{|s-Re(\lambda_i)|^2 + |Im(\lambda_i)|^2} > |s-Re(\lambda_i)| \geq |s-\lambda_n| = s-\lambda_n = \rho(B),$$

which is a contradiction because $s-\lambda_i$ is an eigenvalue of B. On the other hand, if $Re(\lambda_i) < \lambda_n$, then $s - Re(\lambda_i) > s-\lambda_n > 0$. Thus,

$$|s-\lambda_i| \geq |s-Re(\lambda_i)| > |s-\lambda_n| = s-\lambda_n = \rho(B),$$

which is again a contradiction because $s-\lambda_i$ is an eigenvalue of B. Therefore, $\lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$.

Conversely, suppose that $A^{-1} \in WPF_n$ and that $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$. Then, there are semipositive vectors v and w such that $A^{-1}v = \rho(A^{-1})v = \lambda_n^{-1}v$ and $w^T A^{-1} = \rho(A^{-1})w^T = \lambda_n^{-1}w$. Note that for every λ_i such that $|\lambda_i| = \lambda_n$ we have $\lambda_i = \lambda_n$ (otherwise, $0 < \lambda_n < Re(\lambda_i) \leq |\lambda_i| = \lambda_n$, which is a contradiction). Moreover, the set of complex numbers $\{\lambda_i \in \sigma(A) \,:\, |\lambda_i| \neq \lambda_n\} = \sigma(A)\backslash\{\lambda_n\}$ lies completely in the set Ω defined by the intersection of the following two sets:

- The annulus $\{z \,:\, \lambda_n \leq |z| \leq |\lambda_1|\}$, and
- The (open) half-plane $\{z \,:\, Re(z) > Re(\lambda_n)\}$.

It is easy to see that there is a real number s large enough so that the circle centered at s of radius $s-\lambda_n$ surrounds all the complex numbers $\lambda_i \in \sigma(A)$, $\lambda_i \neq \lambda_n$ lying in Ω; see Figure 3.1. For such an s, define the matrix $B_s := sI - A$. Then the eigenvalues of B_s are $s-\lambda_1, s-\lambda_2, \ldots, s-\lambda_n$. Moreover, by our choice of s, we have the following:

$$|s-\lambda_i| < s-\lambda_n \quad \text{for all} \quad \lambda_i \neq \lambda_n.$$

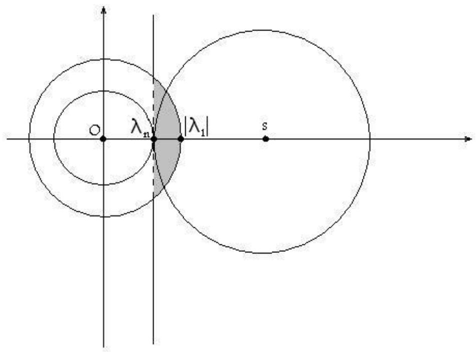

FIG. 3.1. *The gray region represents the set Ω, which is the intersection of the open right half-plane determined by the vertical straight line passing through λ_n and the closed annulus centered at 0 with radii λ_n and $|\lambda_1|$.*

Therefore, $0 < \rho(B_s) = s - \lambda_n < s$. Moreover, $B_s v = (s - \lambda_n)v$ and that $w^T B_s = (s - \lambda_n)w^T$. Thus, $B_s \in WPF_n$. And therefore, $A = sI - B_s$ is a nonsingular GM-matrix. □

In [15, Theorem 8], Johnson and Tarazaga proved that if A is a pseudo-M-matrix, then $A^{-1} \in PF_n$. We extend this theorem by giving necessary and sufficient conditions for a matrix A to be a pseudo-M-matrix. The proof is very similar to that of Theorem 3.1, and thus, it is omitted.

THEOREM 3.2. *Let A be a matrix in $\mathbb{R}^{n\times n}$ whose eigenvalues (when counted with multiplicity) are arranged in the following manner: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Then the following statements are equivalent:*

(i) A is a pseudo-M-matrix.
(ii) A^{-1} exists, A^{-1} is eventually positive, and $0 < \lambda_n < Re(\lambda_i)$, for $i = 1, \ldots, n-1$.

REMARK 3.3. Since every M-matrix is a GM-matrix, it follows that condition (ii) in Theorem 3.1 can be used to check if a matrix is not an inverse M-matrix. In particular, if the real part of any eigenvalue is less than the minimum of all moduli of all eigenvalues then the given matrix is not an inverse M-matrix.

REMARK 3.4. The set WPF_n in Theorem 3.1 is not completely analogous to the set of nonnegative matrices. In other words, if we replace in Theorem 3.1 WPF_n by the set of nonnegative matrices and if we replace a GM-matrix by an M-matrix, then the statement of the theorem would not be correct. Similarly, in Theorem 3.2, PF_n is not completely analogous with the set of positive matrices. For example, we may find a nonnegative matrix whose inverse is a GM-matrix but not an M-matrix. An example of the latter is the positive matrix $C = \frac{1}{36}\begin{bmatrix} 7 & 6 & 5 \\ 5 & 12 & 1 \\ 1 & 6 & 11 \end{bmatrix}$. Note that $C^{-1} = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix} = sI - B$ where $s = 10$, $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \in WPF_3$, and $\rho(B) = 8$. Hence, C^{-1} is a nonsingular GM-matrix. However, C^{-1} is not an M-matrix since it has some positive off-diagonal entries.

Corollary 3.5. *A matrix $C \in \mathbb{R}^{n\times n}$ is an inverse GM-matrix if and only if $C \in WPF_n$ and $Re(\lambda^{-1}) > \rho(C)^{-1}$ for all $\lambda \in \sigma(C)$, $\lambda \neq \rho(C)$.*

Corollary 3.6. *Every real eigenvalue of a nonsingular GM-matrix is positive.*

Example 3.7. In this example, we show a nonsingular EM-matrix whose inverse is not eventually nonnegative. This implies that no result analogous to Theorems 3.1 and 3.2 hold for this case. Let $A = \begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix} = 3I - \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 3I - B$. Then, $\rho(B) = 2$ and, using an induction argument, $B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 0 & 0 \\ 2^{k-1} & 2^{k-1} & 0 & 0 \\ k2^{k-1} & k2^{k-1} & 2^{k-1} & 2^{k-1} \\ k2^{k-1} & k2^{k-1} & 2^{k-1} & 2^{k-1} \end{bmatrix} \geq 0$ for all integers $k \geq 2$. Hence, A is an EM-matrix. But, $A^{-1} = 3^{-2}(E+F)$ where $E = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 9 & 9 & 6 & 3 \\ 9 & 9 & 3 & 6 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Note that $EF = FE = 3F$ and $F^2 = 0$. Therefore, using an induction argument, it is easy to check that $\left(A^{-1}\right)^k = 3^{-2k}E^k + k3^{-k-1}F$. Hence, A^{-1} is not eventually nonnegative because the (1,4) and (2,3) entries are always negative.

It is well-known that a Z-matrix $A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if A is positive stable; see, e.g., [1, p.137]. In the following proposition, we prove an analogous result between GZ-matrices and GM-matrices.

Proposition 3.8. *A GZ-matrix $A \in \mathbb{R}^{n\times n}$ is a nonsingular GM-matrix if and only if A is positive stable.*

Proof. Let A be a GZ-matrix in $\mathbb{R}^{n\times n}$ with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. If A is a nonsingular GM-matrix, then Theorem 3.1 implies that $0 < \lambda_n < Re(\lambda_i)$ for all $\lambda_i \neq \lambda_n$. Thus, $Re(\lambda_i) > 0$ for $i = 1, 2, \ldots, n$. Hence, A is positive stable. Conversely, suppose that A is positive stable, then it follows that 0 is not an eigenvalue of A, which implies that A is nonsingular. Moreover, since A is a GZ-matrix we can decompose A in the following manner $A = sI - B$ where $B \in WPF_n$ and $s \geq 0$. If $s \leq \rho(B)$ then $(s - \rho(B))$ is a nonpositive eigenvalue of A, which contradicts the positive stability of A. Hence, $s > \rho(B)$, which shows that A is a nonsingular GM-matrix. ☐

Another useful result is the following; see, e.g., [1, p.136].

Theorem 3.9. *A Z-matrix $A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if there is a positive vector x such that Ax is positive.*

In Theorem 3.11 below, we prove an analogous result for pseudo-M-matrices. The results in the following lemma are proved in [18, Theorem 2.6].

Lemma 3.10. *If $B \in \mathbb{R}^{n\times n}$ has a left Perron-Frobenius eigenvector and $x = [x_1 \ \cdots \ x_n]^T$ is any positive vector then either $\frac{\sum_{j=1}^n b_{ij}x_j}{x_i} = \rho(B)$ for all $i \in \{1, 2, \ldots, n\}$*

or $min_{i=1}^{n} \frac{\sum_{j=1}^{n} b_{ij}x_j}{x_i} \le \rho(B) \le max_{i=1}^{n} \frac{\sum_{j=1}^{n} b_{ij}x_j}{x_i}$.

THEOREM 3.11. *If $A = sI - B$ where $B \in PF_n$, then the following are equivalent:*

(i) A is a pseudo-M-matrix.

(ii) There is a positive vector x such that Ax is positive.

Proof. Suppose $A = sI - B$ is a pseudo-M-matrix and let x be a right Perron-Frobenius eigenvector of B. Then, $Ax = (sI - B)x = (s - \rho(B))x$ is a positive vector. Conversely, suppose there is a positive vector x such that $Ax = (sI - B)x = sx - Bx$ is positive. Then, $max_{i=1}^{n} \frac{\sum_{j=1}^{n} b_{ij}x_j}{x_i} < s$, and by Lemma 3.10, $\rho(B) \le max_{i=1}^{n} \frac{\sum_{j=1}^{n} b_{ij}x_j}{x_i}$. Hence, $\rho(B) < s$ which proves that A is a pseudo-M-matrix. □

We next give a characterization of nonsingular GM-matrices which has the flavor of Theorem 3.9. We use the following result, which is [6, Lemma 10.7].

LEMMA 3.12. *For any semipositive vector v_1 and for any scalar $\epsilon > 0$, there is an orthogonal matrix Q such that $||Q - I||_2 < \epsilon$ and $Qv_1 > 0$.*

THEOREM 3.13. *If $A = sI - B$ is a GZ-matrix ($B \in WPF_n$), then the following are equivalent:*

(i) A is a nonsingular GM-matrix.

(ii) There is an orthogonal matrix Q such that Qx and QA^Tx are positive where x is a left Perron-Frobenius eigenvector of B.

Proof. Suppose $A = sI - B$ is a nonsingular GM-matrix and let x be a left Perron-Frobenius eigenvector of B. Then, by Lemma 3.12, there is an orthogonal matrix Q such that Qx is positive. Moreover, $QA^Tx = Q(sI - B^T)x = (s - \rho(B))Qx$ is positive since A is nonsingular having $\rho(B) < s$. Hence, $(i) \Rightarrow (ii)$. Conversely, suppose (ii) is true. Then, $QA^Tx = Q(sI - B^T)x = (s - \rho(B))Qx$ is positive, and thus, $\rho(B) < s$. □

We end this section with a result on the classes of an EM-matrix.

PROPOSITION 3.14. *Let $A = sI - B$ be an EM-matrix (B eventually nonnegative and $0 < \rho(B) \le s$). If A is singular, then for every class α of B the following holds:*

1. *$A[\alpha]$ is a singular irreducible EM-matrix if α is basic.*
2. *$A[\alpha]$ is a nonsingular irreducible EM-matrix if α is not basic.*

Proof. If A is a singular EM-matrix and α is a class of B, then $A[\alpha] = sI - B[\alpha]$, where I is the identity matrix having the appropriate dimensions. If α is a basic class of B, then $B[\alpha]$ is an irreducible submatrix of B and $\rho(B[\alpha]) = \rho(B) > 0$. Since the eigenvalues of A are of the form $s - \mu$ where $\mu \in \sigma(B)$ and since A is singular, it follows that $\rho(B) = s$. Hence, $\rho(B[\alpha]) = s$ and $A[\alpha] = sI - B[\alpha]$ must be singular, as well. Moreover, since $B[\alpha]$ is irreducible, it follows that the graph $G(B[\alpha])$ is strongly connected. Note that the graph $G(A[\alpha]) = G(sI - B[\alpha])$ may differ from the graph $G(B[\alpha])$ only in having or missing some loops on some vertices. But this means that the graph $G(A[\alpha])$ is also strongly connected because adding or removing loops from vertices of a strongly connected graph does not affect strong connectedness. Hence, $A[\alpha]$ is irreducible. Moreover, if $\kappa = (\alpha_1, \ldots, \alpha_m)$ is an ordered partition of $\{1, 2, \ldots, n\}$ that gives the Frobenius normal form of B (see, e.g., [2]), then B_κ is block triangular and it is permutationally similar to B. Thus, B_κ is eventually nonnegative and so is each of its diagonal blocks. In particular, there is a diagonal block in B_κ which is permutationally similar to $B[\alpha]$ (because α is a class of B). Hence, $B[\alpha]$ is eventually nonnegative, which proves part 1. Similarly, if α is not a basic class of B, then part 2 holds. □

4. Splittings and GM-matrices. In this section, we define various splittings of a GM-matrix, give sufficient conditions for convergence, and give several examples.

Recall that a splitting of a matrix A is an expression of the form $A = M - N$ where M is nonsingular. We begin by listing some preliminary definitions.

DEFINITION 4.1. *Let $A = M - N$ be a splitting. Then, such a splitting is called*
- *weak (or nonnegative) if $M^{-1}N \geq 0$.*
- *weak-regular if $M^{-1}N \geq 0$ and $M^{-1} \geq 0$ [19].*
- *regular if $M^{-1} \geq 0$ and $N \geq 0$ [23].*
- *M-splitting if M is an M-matrix and $N \geq 0$ [21].*
- *Perron-Frobenius splitting if $M^{-1}N$ possesses the Perron-Frobenius property [18].*

We list now the new splittings introduced in this paper. We begin first by defining the splitting having the Perron singular property, which is a splitting for an arbitrary nonsingular matrix. Then we proceed to define the splittings specific to nonsingular GM-matrices.

DEFINITION 4.2. *Let A be nonsingular. We say that the splitting $A = M - N$ has the Perron singular property if $\gamma M + (1-\gamma)N$ is singular for some $\gamma \in \mathbb{C}$, $\gamma \neq 0$ and $M^{-1}N$ has the Perron-Frobenius property.*

Note that a splitting with the Perron singular property is, in particular, a Perron-Frobenius splitting.

DEFINITION 4.3. *Let $A = M - N$ be a splitting of a nonsingular GM-matrix $A = sI - B$ ($B \in WPF_n$ and $\rho(B) < s$). Then, such a splitting is called*
- *a G-regular splitting if M^{-1} and N are in WPF_n.*
- *a GM-splitting if M is a GM-matrix and $N \in WPF_n$.*
- *an overlapping splitting if $E_\lambda(M^{-1}N) \cap E_{\rho(B)}(B) \neq \{0\}$ with $|\lambda| = \rho(M^{-1}N)$.*
- *a commuting bounded splitting if M and N commute and $\rho(M) < s$.*

REMARK 4.4. A GM-splitting is a G-regular splitting but not conversely. For example, consider the GM-matrix $A = diag(1,4,4) = sI - B$ where $s = 5$ and $B = diag(4,1,1)$. An example of a G-regular splitting of A is the splitting $A = M - N$ where $M = diag(2,32,-4)$ and $N = diag(1,28,-8)$. Note that M^{-1} is in WPF_n yet, by Theorem 3.1, M is not a GM-matrix. Hence, this G-regular splitting is not a GM-splitting.

LEMMA 4.5. *Let $A = M - N$ be a splitting of a nonsingular matrix A. Then, the following are equivalent:*

(i) The splitting is convergent.

(ii) $min\ \{Re(\lambda) \mid \lambda \in \sigma(NA^{-1})\} > -\frac{1}{2}$.

(iii) $min\ \{Re(\lambda) \mid \lambda \in \sigma(A^{-1}N)\} > -\frac{1}{2}$.

Proof. We prove first the equivalence of (i) and (ii). Let $P = M^{-1}NA^{-1}M$. Thus, P and NA^{-1} are similar matrices, and therefore, they have the same eigenvalues with the same multiplicities. Moreover, the following relation between P and $M^{-1}N$ holds:

$$P = M^{-1}NA^{-1}M = M^{-1}N(M-N)^{-1}M = M^{-1}N(I - M^{-1}N)^{-1}.$$

Hence, the eigenvalues of NA^{-1} and $M^{-1}N$ are related as follows: $\mu \in \sigma(M^{-1}N)$ if and only if there is a unique $\lambda \in \sigma(NA^{-1})$ such that $\mu = \frac{\lambda}{1+\lambda}$. The splitting is convergent, i.e., $\rho(M^{-1}N) < 1$ if and only if for all $\mu \in \sigma(M^{-1}N)$, we have $|\mu| < 1$. That is, if for all $\lambda \in \sigma(NA^{-1})$, we have $\left|\frac{\lambda}{1+\lambda}\right| < 1$, or equivalently, $\frac{(Re(\lambda))^2 + (Im(\lambda))^2}{(1+Re(\lambda))^2 + (Im(\lambda))^2} < 1$, which holds only whenever $2Re(\lambda) + 1 > 0$, or whenever (ii) is true. As for the equivalence of (i) and (iii), it follows similarly by noting the following relation between $A^{-1}N$ and $M^{-1}N$:

$$A^{-1}N = (M-N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N. \qquad \square$$

COROLLARY 4.6. *Let $A = M - N$ be a splitting of a nonsingular matrix A. If $A^{-1}N$ or NA^{-1} is an inverse GM-matrix, then the splitting is convergent.*

Proof. Let P denote $A^{-1}N$ or NA^{-1}. If P is an inverse GM-matrix then, by Corollary 3.5, $Re(\lambda^{-1}) > (\rho(P))^{-1} > 0$ for all $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. This implies that $Re(\lambda) = |\lambda|^2 Re(\lambda^{-1}) > -\frac{1}{2}$ for all $\lambda \in \sigma(P)$, $\lambda \neq \rho(P)$. Thus, $Re(\lambda) > -\frac{1}{2}$ for all $\lambda \in \sigma(P)$, which is equivalent to condition (ii) of Lemma 4.5 if $P = NA^{-1}$, or equivalent to condition (iii) of Lemma 4.5 if $P = A^{-1}N$. Hence, the given splitting is convergent. $\square$

The following lemma is Theorem 2.3 of [18] and Lemma 5.1 of [6].

LEMMA 4.7. *Let A be a nonnilpotent eventually nonnegative matrix. Then, both A and A^T possess the Perron-Frobenius property, i.e., $A \in WPF_n$.*

THEOREM 4.8. *If $A = M - N$ is a splitting having the Perron singular property, then any of the following conditions is sufficient for convergence:*

(A1) $A^{-1}N$ *is eventually positive.*
(A2) $A^{-1}N$ *is eventually nonnegative.*
(A3) $A^{-1}N \in WPF_n$.
(A4) $A^{-1}N$ *has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*
(A5) $A^{-1}N$ *has a basic and an initial class α such that $(A^{-1}N)[\alpha]$ has a right Perron-Frobenius eigenvector.*

Proof. We prove first $(A2) \Rightarrow (A3) \Rightarrow$ convergence of the given splitting. Suppose that $A^{-1}N$ is eventually nonnegative. Since $A = M - N$ is a splitting having the Perron singular property, it follows that there is a nonzero complex scalar γ such that $\gamma M + (1-\gamma)N$ is singular. Hence, $\gamma A + N$ is singular $\Leftrightarrow det(\gamma A + N) = 0 \Leftrightarrow det(-\gamma I - A^{-1}N) = 0$. In other words, $-\gamma$ is a nonzero eigenvalue of $A^{-1}N$, i.e., it is not nilpotent. By Lemma 4.7, $A^{-1}N$ and its transpose possess the Perron-Frobenius property, i.e., $A^{-1}N \in WPF_n$. And thus, the given splitting converges by Lemma 4.11. As for the rest of the sufficient conditions, we outline the proof using the following diagram:

$$\begin{array}{ccccccc} (A1) & \Rightarrow & (A2) & \Rightarrow & (A3) & \Rightarrow & \text{convergence} \\ \Updownarrow & & & & \Uparrow & & \\ (A4) & & & & (A5) & & \end{array}$$

The equivalency and implications in the above diagram follow from the results in [6] on eventually positive matrices, eventually nonnegative matrices, and matrices in WPF_n. $\square$

REMARK 4.9. Recall that a regular splitting $A = M - N$ of a monotone matrix (i.e., when $A^{-1} \geq 0$) is convergent [23]. Thus, Theorem 4.8 is a genaralization of this situation since we do not require that A^{-1} nor N, nor their product $A^{-1}N$ be nonnegative.

EXAMPLE 4.10. Let $A = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ and consider the splitting $A = M - N$, where $M = \frac{1}{4}\begin{bmatrix} 29 & -6 & -11 \\ -11 & 18 & 5 \\ 5 & -6 & 13 \end{bmatrix}$ and $N = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. For $\gamma = -\frac{1}{2}$

the matrix $\gamma M + (1-\gamma)N = \frac{1}{4}\begin{bmatrix} -13 & 6 & 7 \\ 7 & -6 & -1 \\ -1 & 6 & -5 \end{bmatrix}$ is singular. Moreover, $M^{-1}N = \frac{1}{12}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ is a positive matrix and thus it possesses the Perron-Frobenius property. Hence, this splitting is a splitting with the Perron singular property. Since $A^{-1}N = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ is a positive matrix (and hence eventually positive), it follows from Theorem 4.8 that this splitting is convergent. In fact, $\rho(M^{-1}N) = \frac{1}{3} < 1$.

The following lemma is part of Theorem 3.1 of [18].

LEMMA 4.11. *Let $A = M - N$ be a Perron-Frobenius splitting of a nonsingular matrix $A \in \mathbb{R}^{n\times n}$. Then, the following are equivalent:*

(i) $\rho(M^{-1}N) < 1$.

(ii) $A^{-1}N$ *possesses the Perron-Frobenius property.*

(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$.

COROLLARY 4.12. *Let $A = M - N$ be a splitting of a nonsingular matrix A such that N is nonsingular and $N^{-1}M$ is a nonsingular GM-matrix. Then, the following are equivalent:*

(i) The splitting $A = M - N$ is convergent.

(ii) $A^{-1}N$ *possesses the Perron-Frobenius property.*

(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$.

Proof. Since $N^{-1}M$ is a nonsingular GM-matrix, it follows that $(N^{-1}M)^{-1} = M^{-1}N \in \mathrm{WPF}_n$. Hence, $M^{-1}N$ satisfies the Perron-Frobenius property, which implies that the splitting $A = M - N$ is a Perron-Frobenius splitting and the equivalence of the statements in the corollary follows from Lemma 4.11. ◻

PROPOSITION 4.13. *If $A = sI - B$ is a GM-matrix and the splitting $A = M - N$ is an overlapping splitting (for which $E_\lambda(M^{-1}N) \cap E_{\rho(B)}(B) \neq \{0\}$ and $|\lambda| = \rho(M^{-1}N)$), then such a splitting is convergent if and only if $\exists\ \eta = \frac{s-\rho(B)}{1-\lambda} \in \sigma(M)$ such that $Re(\eta) > \frac{s-\rho(B)}{2}$.*

Proof. Note first that if $A = sI - B = M - N$ is an overlapping splitting then we can pick a semipositive vector $v \in E_{\rho(B)}(B) \cap E_\lambda(M^{-1}N)$. And for this vector, we have:

$$
\begin{aligned}
&(sI - B)v \ = Av = (M-N)v = M^{-1}(I - M^{-1}N)v \\
\Leftrightarrow\ & (s-\rho(B))Mv = (I - M^{-1}N)v = (1-\lambda)v \\
\Leftrightarrow\ & Mv = \frac{(1-\lambda)}{(s-\rho(B))}v \\
\Rightarrow\ & \exists\ \eta \in \sigma(M) \ni \eta = \frac{(1-\lambda)}{(s-\rho(B))} \\
\Leftrightarrow\ & \exists\ \eta \in \sigma(M) \ni \lambda = \frac{\eta - (s-\rho(B))}{\eta}.
\end{aligned}
$$

Hence, if $A = M - N$ is an overlapping splitting then there is an eigenvalue $\eta \in \sigma(M)$ such that $\lambda = \frac{\eta-(s-\rho(B))}{\eta}$. Therefore, an overlapping splitting is convergent, i.e., $\rho(M^{-1}N) = |\lambda| < 1$, when $|\eta - (s-\rho(B))| < |\eta|$ for some $\eta \in \sigma(M)$, or equivalently whenever η lies in the right-half plane determined by the perpendicular bisector of the segment on the real axis whose endpoints are 0 and $(s-\rho(B))$, i.e., whenever $Re(\eta) > \frac{s-\rho(B)}{2}$. □

COROLLARY 4.14. *Let $A = M - N$ be an overlapping splitting of a nonsingular GM-matrix A and suppose that $M^{-1}N \in WPF_n$. If $\frac{s-\rho(B)}{1-\rho(M^{-1}N)} \in \sigma(M)$ then $\rho(M^{-1}N) < 1$, i.e. the splitting is convergent.*

EXAMPLE 4.15. Let A be as in Example 4.10. Then, A is a nonsingular GM-matrix. In fact, $A = sI - B$, where $s = 10$, $B = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \in WPF_3$, and $\rho(B) = 8$. An overlapping splitting of the matrix A is $A = M - N$ where $M = \frac{1}{8}\begin{bmatrix} 55 & -18 & -25 \\ -25 & 30 & 7 \\ 7 & -18 & 23 \end{bmatrix}$ and $N = \frac{1}{8}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$. Note that $M^{-1}N = \frac{2}{3}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ and that for $\lambda = -\frac{1}{3} \in \sigma(M^{-1}N)$ we have $|\lambda| = \frac{1}{3} = \rho(M^{-1}N)$ and $E_\lambda(M^{-1}N) = E_{\rho(B)}(B) = Span\{[1\ 1\ 1]^T\}$. Hence, this overlapping splitting is convergent. Proposition 4.13 predicts the existence of an eigenvalue η of M such that $\eta = \frac{s-\rho(B)}{1-\lambda} = \frac{10-8}{1-(-1/3)} = \frac{3}{2}$ and $Re(\eta) = \frac{3}{2} > \frac{s-\rho(B)}{2} = 1$. If we look at the spectrum of M we see that $\frac{3}{2} \in \sigma(M) = \left\{\frac{3}{2}, 6\right\}$ just as predicted by Proposition 4.13. On the other hand, if $A = M - N$ where $M = \frac{1}{4}\begin{bmatrix} 27 & -10 & -13 \\ -13 & 14 & 3 \\ 3 & -10 & 11 \end{bmatrix}$ and $N = \frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ then $M^{-1}N = N = \frac{1}{4}\begin{bmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix}$ and for $\lambda = -1 \in \sigma(M^{-1}N)$ we have $|\lambda| = 1 = \rho(M^{-1}N)$ and $E_\lambda(M^{-1}N) = E_{\rho(B)}(B) = Span\{[1\ 1\ 1]^T\}$. Hence, the latter splitting is an overlapping splitting but it does not converge. Proposition 4.13 predicts that for all $\eta \in \sigma(M)$ either $\eta \neq \frac{s-\rho(B)}{1-\lambda} = \frac{10-8}{1-(-1)} = 1$ or $Re(\eta) \leq \frac{s-\rho(B)}{2} = 1$, which is true about the spectrum of M since $\sigma(M) = \{1, 6\}$.

THEOREM 4.16. *A GM-matrix $A = sI - B$ having a commuting bounded splitting $A = M - N$ induces a splitting of B of the form $B = M' - N'$ where $M' = \frac{1}{\omega}(sI - M)$, $\omega \in \mathbb{R}$ and $\omega \neq 0$. Moreover, if $|\omega| < min \left\{\frac{s}{\rho(M')}, \frac{1}{2\rho(M')}\right\}$, then the commuting bounded splitting is convergent.*

Proof. Suppose that the GM-matrix $A = sI - B$ has a commuting bounded splitting $A = M - N$ and let $M' = \frac{1}{\omega}(sI - M)$ and $N' = A - sI + M'$ for some $\omega \in \mathbb{R}$ and $\omega \neq 0$. Then, M' is nonsingular because $\rho(M) < s$ and $M = sI - \omega M'$. Moreover, we can write $A = (sI - \omega M') - ((1-\omega)M' - N')$. Note that the iteration matrix of the commuting bounded splitting of A is $M^{-1}N = (sI - \omega M')^{-1}((1-\omega)M' - N')$. Since M and N commute, so do M' and N'. Furthermore, there is a single unitary matrix U that produces the Schur decomposition (see, e.g., [11, p.81]) of both M' and N'. Hence, an eigenvalue of $M^{-1}N$ would have the form $(s-\omega\lambda)^{-1}((1-\omega)\lambda - \mu)$ where $\lambda \in \sigma(M')$ and $\mu \in \sigma(N')$. But, since M' and N' are simultaneously Schur decomposable, it follows that the same unitary matrix that produces the Schur decomposition of M' and N' also produces the Schur decomposition of B and A. Therefore, $\lambda - \mu$ is an eigenvalue of B which does not exceed s in modulus (since A is a GM-matrix). Thus,

$$\left|(s-\omega\lambda)^{-1}((1-\omega)\lambda-\mu)\right| = \frac{|(\lambda-\mu)-\omega\lambda|}{|s-\omega\lambda|} \leq \frac{|\lambda-\mu|+|\omega||\lambda|}{|s-\omega\lambda|} \leq \frac{s+|\omega|\rho(M')}{|s-\omega\lambda|}$$

Moreover, if we choose $|\omega| < \dfrac{s}{\rho(M')}$, then $s-|\omega|\rho(M') > 0$. Hence,

$$\left|(s-\omega\lambda)^{-1}((1-\omega)\lambda-\mu)\right| \leq \frac{s+|\omega|\rho(M')}{|s-\omega\lambda|} \leq \frac{s+|\omega|\rho(M')}{s-|\omega|\rho(M')}.$$

Therefore, if

$$\frac{s+|\omega|\rho(M')}{s-|\omega|\rho(M')} < 1 \tag{4.1}$$

then the splitting $A = M - N$ is convergent. But (4.1) is equivalent to $|\omega| < \frac{1}{2\rho(M')}$. Hence, if $|\omega| < min \left\{\frac{s}{\rho(M')}, \frac{1}{2\rho(M')}\right\}$, then the splitting $A = M - N$ is convergent. □

EXAMPLE 4.17. Let $A = \frac{1}{40}\begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 0 \\ 1 & 1 & 4 \end{bmatrix}$. Then, $A = sI - B$ is a GM-matrix, where $s = 1$, $B = \frac{1}{40}\begin{bmatrix} 37 & -1 & -2 \\ -1 & 35 & 0 \\ -1 & -1 & 36 \end{bmatrix} \in WPF_3$, and $\rho(B) = 0.95$. A commuting bounded splitting of A is $A = M - N$, where $M = \frac{1}{80}\begin{bmatrix} 73 & 2 & 1 \\ 1 & 74 & 1 \\ 1 & 2 & 73 \end{bmatrix}$ and $N = \frac{1}{80}\begin{bmatrix} 67 & 0 & -3 \\ -1 & 64 & 1 \\ -1 & 0 & 65 \end{bmatrix}$. Note that $\rho(M) = 0.95 < 1 = s$ and that $MN = NM = \frac{1}{1600}\begin{bmatrix} 1222 & 32 & -38 \\ -2 & 1184 & 34 \\ -2 & 32 & 1186 \end{bmatrix}$. Furthermore, let $\omega = 5$ and let $M' = \frac{1}{\omega}(sI - M) = \frac{1}{400}\begin{bmatrix} 7 & -2 & -1 \\ -1 & 6 & -1 \\ -1 & -2 & 7 \end{bmatrix}$. Then, $\rho(M') = 0.02$ making $|\omega| = 5 < min\left\{\frac{s}{\rho(M')}, \frac{1}{2\rho(M')}\right\} = min\left\{\frac{1}{0.02}, \frac{1}{2(0.02)}\right\} = 12.5$. Hence, by Theorem 4.16, the splitting $A = M - N$ is convergent. In fact, $\rho(M^{-1}N) \approx 0.9444 < 1$.

THEOREM 4.18. *If $A = M - N$ is a splitting of a GM-matrix A, then any Type I condition (listed below) implies that such a splitting is a G-regular splitting. Moreover, if the splitting $A = M - N$ is a G-regular splitting that satisfies one of Type II conditions (listed below), then any one of Type III conditions (listed below) is sufficient for convergence.*

Type I Conditions

- *(D1) M^{-1} and N are eventually positive.*
- *(D2) M^{-1} and N are eventually nonnegative with N not being nilpotent.*
- *(D3) Each of M^{-1} and N has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*

(D4) *Each of* M^{-1} *and* N *has two classes* α *and* β*, not necessarily distinct, such that:*
  (i) α *is basic, initial, and* $X[\alpha]$ *has a right Perron-Frobenius eigenvector.*
  (ii) β *is basic, final, and* $X[\beta]$ *has a left Perron-Frobenius eigenvector.*

Type II Conditions

(E1) $M^{-1}N$ *is eventually positive.*
(E2) $M^{-1}N$ *is eventually nonnegative but not nilpotent.*
(E3) $M^{-1}N \in WPF_n$.
(E4) $M^{-1}N$ *has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*
(E5) $M^{-1}N$ *has a basic and an initial class* α *such that* $(M^{-1}N)[\alpha]$ *has a right Perron-Frobenius eigenvector.*

Type III Conditions

(F1) $A^{-1}N$ *is eventually positive.*
(F2) $A^{-1}N$ *is eventually nonnegative but not nilpotent.*
(F3) $A^{-1}N \in WPF_n$.
(F4) $A^{-1}N$ *has a simple positive and strictly dominant eigenvalue with a positive spectral projector of rank 1.*
(F5) $A^{-1}N$ *has a basic and an initial class* α *such that* $(A^{-1}N)[\alpha]$ *has a right Perron-Frobenius eigenvector.*

Proof. We prove the theorem for the following Type I, Type II, and Type III conditions, respectively: (D1), (E1), and (F1), and then we outline the rest of the proof. Suppose that the splitting $A = M - N$ satisfies condition (D1). Then, (D1) is true if and only if M^{-1} and N are in $PF_n \subset WPF_n$. Hence, the splitting $A = M - N$ is a G-regular splitting. Moreover, suppose that $A = M - N$ is a G-regular splitting and that (E1) is true. Then, $M^{-1}N \in PF_n$ and thus $M^{-1}N$ has the Perron-Frobenius property. In particular, the given G-regular splitting becomes a Perron-Frobenius splitting. If (F1) is true then $A^{-1}N \in PF_n$ and thus $A^{-1}N$ possesses the Perron-Frobenius property. Hence, by Lemma 4.11, the G-regular splitting converges. With regards to the remaining conditions, we use the following diagrams to outline the proof:

$$\begin{array}{ccccccc}
(D1) & \Rightarrow & (D2) & \Rightarrow & M^{-1}, N \in WPF_n & \Leftrightarrow & A = M - N \text{ is a } G\text{-regular splitting} \\
\Updownarrow & & & & \Uparrow & & \\
(D3) & & & & (D4) & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & & & & (E5) \\
 & & & & & & \Downarrow \\
(E1) & \Rightarrow & (E2) & \Rightarrow & (E3) & \Rightarrow & M^{-1}N \text{ has the Perron-Frobenius property} \\
\Updownarrow & & & & & & \Downarrow \\
(E4) & & & & & & A = M - N \text{ is a Perron-Frobenius splitting} \\
 & & & & & & \text{as well as a } G\text{-regular splitting}
\end{array}$$

$$\begin{array}{ccccccc}
 & & & & & & (F5) \\
 & & & & & & \Downarrow \\
(F1) & \Rightarrow & (F2) & \Rightarrow & (F3) & \Rightarrow & A^{-1}N \text{ has the Perron-Frobenius property} \\
\Updownarrow & & & & & & \Downarrow \\
(F4) & & & & & & \text{The splitting converges by Lemma 4.11}
\end{array}$$

All the above implications and equivalencies follow from the results in [6] on eventually nonnegative matrices, eventually positive matrices, and matrices in WPF_n. □

EXAMPLE 4.19. Let $A = \begin{bmatrix} 8 & -3 & -4 \\ -4 & 4 & 0 \\ 0 & -3 & 3 \end{bmatrix} = 8I - B = 8I - \begin{bmatrix} 0 & 3 & 4 \\ 4 & 4 & 0 \\ 0 & 3 & 5 \end{bmatrix}$. Then, A is a nonsingular GM-matrix and $\rho(B) = \frac{1}{2}(7 + \sqrt{73}) \approx 7.7720 < 8$. Consider the splitting $A = M - N$ where $M = \begin{bmatrix} 7 & -2 & -3 \\ -3 & 4 & 1 \\ 1 & -2 & 3 \end{bmatrix}$ (a nonsingular GM-matrix from the previous examples) and $N = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in WPF_3$. Thus, this splitting of A is a GM-splitting, and hence, a G-regular splitting. Note that $M^{-1}N = \frac{1}{36}\begin{bmatrix} 4 & 12 & 13 \\ 8 & 6 & 17 \\ 16 & 12 & 7 \end{bmatrix}$ is an eventually positive matrix, a Type II condition in Theorem 4.18. Moreover, $A^{-1}N = \frac{1}{12}\begin{bmatrix} 25 & 28 & 33 \\ 28 & 28 & 36 \\ 32 & 32 & 36 \end{bmatrix}$ is an eventually positive matrix, a Type III condition in Theorem 4.18. Hence, Theorem 4.18 predicts the convergence of this G-regular splitting. In fact, $\rho(M^{-1}N) \approx 0.8859 < 1$.

Acknowledgement. We thank Dimitrios Noutsos for his comments on an earlier version of this paper.

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