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ON MESH INDEPENDENCE OF CONVERGENCE BOUNDS FOR ADDITIVE SCHWARZ PRECONDITIONED GMRES*

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Abstract. Additive Schwarz preconditioners, when including a coarse grid correction, are said to be optimal for certain discretized partial differential equations, in the sense that bounds on the convergence of iterative methods are independent of the mesh size h. Cai and Zou [*Numer. Linear Algebra Appl.*, 9:379–397, 2002] showed with a one-dimensional example that in the absence of a coarse grid correction the usual GMRES bound has a factor of the order of $1/\sqrt{h}$. In this paper we consider the same example and show that for that example the behavior of the method is not well represented by the above mentioned bound: We use an a posteriori bound for GMRES from [Simoncini and Szyld, *SIAM Rev.*, 47:247–272, 2005] and show that for that example a relevant factor is bounded by a constant. Furthermore, for a sequence of meshes, the convergence curves for that one-dimensional example, and for several two-dimensional model problems, are very close to each other, and thus the number of preconditioned GMRES iterations needed for convergence for a prescribed tolerance remains almost constant.

Key words. Linear systems, additive Schwarz Preconditioning, GMRES, discretized differential equations, convergence dependence on mesh size

AMS subject classifications. 65F10, 65M99, 65N22.

1. Introduction.

We discuss here some aspects of the convergence of the GMRES [7] iterative method, when it is applied to certain discretized partial differential equations (PDEs), and preconditioned with an additive Schwarz preconditioner with no coarse grid correction. The application of the additive Schwarz preconditioner is obtained by solving several small linear systems, and it is easily parallelizable; see section 3 for a brief description, including the concept of a coarse grid correction, and also, e.g., the monographs [10], [12] for further details.

This paper was inspired by the work of Cai and Zou [1]. They presented a simple discretized one-dimensional PDE, for which a standard bound of GMRES for the problem preconditioned with additive Schwarz with no coarse grid correction (and with fixed overlap) has a factor of order $1/\sqrt{h}$, where h is the mesh size used in the discretization. In other words, Cai and Zou warn that additive Schwarz preconditioners may not be optimal using the standard GMRES (minimizing the l^2 norm); for a discussion of possibly using a version of GMRES minimizing other norms, see [15].

While the above-mentioned bound does indeed depend on h, we show here that the convergence of GMRES is independent of the mesh size. We prove that for the same one-dimensional problem from [1], a factor in an *a posteriori* bound for GMRES convergence given in [8] is constant. We present computational experiments, where we observe that the convergence curves of GMRES for varying values of h are very close to each other, and therefore the number of iterations to converge below a prescribed tolerance is pretty constant, i.e., independent of the value of h. These numerical observations on GMRES convergence independent of the mesh size, are also obtained for several (nonsymmetric) two-dimensional model problems preconditioned with additive Schwarz with no coarse grid correction. Furthermore, the main factor

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in the *a posteriori* bound mentioned above is also pretty constant when varying h, consistent with the observations on the convergence just mentioned.

We should mention that while it holds for the model problems discussed here, that the convergence is independent of the mesh size h, the convergence does depend on the number of subdomains, or what is the same, on the width H of the subdomains. Widlund [14] has shown that for elliptic problems on the plane, the number of iterations of Krylov methods preconditioned with additive Schwarz with no coarse grid correction may grow at a rate of order of $1/H$; see also [12]. We note also that, as is well known, adding a course grid correction to additive Schwarz produces a preconditioned problems whose convergence can be bound independently of the mesh size and the number of subdomains; see, e.g., [10],[12].

The paper is organized as follows. In section 2 we describe the one-dimensional problem from [1], as well as several nonsymmetric two-dimensional counterparts. As already mentioned, in section 3 we briefly describe the additive Schwarz preconditioner, and in section 4 we review a standard convergence bound for GMRES, as well as an *a posteriori* bound. In section 5 analysis and experiments are presented for the one-dimensional problem from [1], while in section 6 we report several numerical experiments for the various two-dimensional model problems, and we end the paper with some concluding remarks. We mention that some preliminary results related to this paper were first reported in [2].

2. The model problems. Our model problems are elliptic PDEs with Dirichlet boundary conditions in one and two dimensions. In one dimensions we consider the unit interval $[0,1]$, and in two dimensions the unit square $[0,1]\times[0,1]$. The one-dimensional problem considered in [1] is

$$\begin{aligned} u_{xx} &= f \qquad \text{in } \Omega = [0,1] \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The two-dimensional model problems we consider are of the form

$$\begin{aligned} -\Delta u + au + bu_x &= f \qquad \text{in } \Omega = [0,1]\times[0,1], \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

for the following three configuration of the constant parameters a and b:

$$a = 0;\ b = 0, \tag{2.3}$$
$$a \neq 0;\ b = 0, \tag{2.4}$$
$$a \neq 0;\ b \neq 0. \tag{2.5}$$

The problems (2.1), (2.2) are discretized using finite elements with linear basis functions as described, e.g., in [3]. The mesh size h is the size of the uniform discretization of Ω. Thus, in the one-dimensional case, $[0,1]$ is divided into $n+1$ intervals, so that $h = 1/(n+1)$, with $N = n$ interior mesh points. Similarly, for the two-dimensional model problem (2.2), let $h = 1/(n+1)$ be the mesh size, and we use an uniform mesh for $\Omega = [0,1]\times[0,1]$, thus having $N = n^2$ interior mesh points.

In our experiments reported later in the paper in section 6, we have chosen f in (2.1) as random, while for the two-dimensional case, f is chosen, so that the exact solution of (2.2) is known. Let

$$Ax = b \tag{2.6}$$

be the linear system obtained from discretizing either (2.1) or (2.2). The right hand side b of (2.6) is the discretization of f in each case. When a (left) preconditioner M is used, one solves a linear system of equations of the form

$$M^{-1}Ax = M^{-1}b, \tag{2.7}$$

instead of (2.6).

3. Additive Schwarz preconditioner. We briefly describe the additive Schwarz preconditioning; see, e.g., [10], [12], for full details. In [1], the problem (2.1) was solved in $\Omega = [0, 1]$, subdivided into two overlapping subdomains $\Omega_1 = [0, l_2]$ and $\Omega_2 = [l_1, 1]$, where $0 < l_1 < l_2 < 1$; see Figure 3.1.

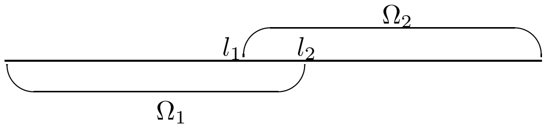

FIG. 3.1. *Domain decomposition for the one-dimensional model problem* (2.1)

In general, one has a subdivision of the domain Ω into s overlapping subdomains Ω_i, $i = 1, \ldots, s$. For each subdomain, one has a corresponding subset of the finite element function space with support in the subdomain. Equivalently, for each subdomain, there is a subspace of V_i of $V = \mathbb{R}^N$, corresponding to the nodal values in Ω_i. Let R_i be the restriction operator from V to V_i, which is an $n_i \times n$ matrix formed by rows of the $N \times N$ identity matrix, where n_i is the number of nodes in subdomain Ω_i. Let $A_i = R_i A R_i^T$, defining a restriction of A to V_i, Then, the additive Schwarz preconditioner

$$M^{-1} = \sum_{i=1}^{s} R_i^T A_i^{-1} R_i \tag{3.1}$$

Consider an additional subspace V_0, corresponding to some functions with support in each of the subdomains Ω_i, $i = 1, \ldots, s$. With R_0 the restriction operator from V to V_0, a coarse grid correction is $R_0^T A_0^{-1} R_0$, with $A_0 = R_0 A R_0^T$. The additive Schwarz preconditioner with a coarse grid correction is then $\sum_{i=0}^{s} R_i^T A_i^{-1} R_i$, i.e., it is similar to (3.1) with the sum starting with $i = 0$.

4. Convergence bounds. Let us consider the solution of the preconditioned problem (2.7) using GMRES. Let x_0 be an initial vector for its solution, let x_k be the approximation at the kth iteration, and $r_k = b - Ax_k$ the corresponding residual. Let $B = M^{-1}A$, then $r_k \in BK_m(B, r_0) := \mathrm{span}\{Br_0, B^2 r_0, \ldots, B^m r_0\}$. Here, and throughout the paper all norms are l^2 norms. Assuming that $M^{-1}A$ is diagonalizable, one standard convergence bound for GMRES is as follows:

$$\|r_k\| \le \kappa(Q)\epsilon^{(k)}\|r_0\|, \tag{4.1}$$

where Q is the eigenvector matrix of $M^{-1}A$, $\kappa(Q) = \|Q\|\|Q^{-1}\|$ is its condition number, and $\epsilon^{(k)}$ is given by

$$\epsilon^{(k)} = \min_{p \in \mathcal{P}_k} \max_{i=1,\ldots,n} |p(\lambda_i)|,$$

where $\mathcal{P}_k$ is the space of the polynomials p of degree less than or equal to k such that $p(0) = 1$, and λ_i are the eigenvalues of $M^{-1}A$; see, e.g., [5], [6], [7], [9], for descriptions of GMRES and the derivation of (4.1). It follows that the bound (4.1) can be very large if the eigenvector matrix Q is not well conditioned. Later, in Figure 4.1 we illustrate how pessimistic this bound can be sometimes.

The convergence behavior of GMRES is, in most cases, superlinear, i.e., the residual norm decreases linearly for a while, and this rate of decrease accelerates as the iterations progress; see, e.g., [8] and references therein. As it turns out, the convergence behavior of GMRES, may depend in part in each of the periods of linear decline on only a few eigenvectors of the relevant matrix $M^{-1}A$, and not on the whole set, i.e., on all columns of the matrix Q; see [8], [9]. Let us write $Q = [Q_1, Q_0]$, let $T = Q^{-*} = [T_1, T_0]$, and let $P_{Q_1} = Q_1 T_1^*$ be the spectral projector onto $\mathcal{R}(Q_1)$, the range of Q_1, where the symbol $*$ stands for conjugate transpose, and $-*$ for its inverse. For details of spectral projections, see, e.g., [13]. Let Π_X denote the orthogonal projection onto the range of the matrix X, and let Y be a matrix whose columns are a basis of a k-dimensional subspace of $BK_m(B, r_0)$. It was shown in [8] that the following holds

$$\|r_{m+j}\| \leq \min_{d \in BK_j(B,r_m)} \{\|(I - P_{Q_1})(r_m - d)\| + \gamma \|P_{Q_1}(r_m - d)\|\}, \tag{4.2}$$

where

$$\gamma = \|(I - \Pi_Y) P_{Q_1}\| \leq \|\Pi_{Q_1} - \Pi_Y\| \|P_{Q_1}\|. \tag{4.3}$$

The bound (4.2) indicates that the behavior of GMRES at after the mth iteration, resembles (except for a factor that depends on γ) that of a GMRES iteration which starts with an initial vector $(I - P_{Q_1})r_m$, i.e., the current residual which has been stripped of its components in the subspace generated by the columns of Q_1 (the selected eigenvectors of $M^{-1}A$). Let us call $\bar{r}_j$ the residual of this GMRES process.

The quantity $\|\Pi_{Q_1} - \Pi_Y\|$ which is always less than one, is sometimes called the gap between the two subspaces $\mathcal{R}(Q_1)$ and $\mathcal{R}(Y) \subset BK_j(B, r_0)$ (see, e.g, [13]), in other words, this quantity depends on how well the Krylov subspace $BK_j(B, r_0)$ approximates the invariant subspace of $M^{-1}A$ generated by the columns of Q_1.

As it can be appreciated, $\|P_{Q_1}\|$ is thus the main factor in the bound (4.3) of the quantity γ used in (4.2). Also, it should be noted that $\|P_{Q_1}\| = \|I - P_{Q_1}\|$, so that this quantity is also present in any bound of (4.2); see, e.g., [11] for proofs of this latter identity. We mention here is that the bound (4.2) is valid for any choice of Q_1, a subset of columns of Q. In fact, there is total flexibility in the choice of this Q_1; and we take advantage of this fact in our experiments. We refer the reader to [8] for other bounds of (4.2) and further considerations.

We remark that we do not advocate the use of the bound (4.2) for analysis of mesh independence of Schwarz or other preconditioners in general. The point we want to stress is that even if some eigenvectors have a dependency on the mesh size, the convergence of GMRES may be completely independent of that parameters. This follows from the fact that only a few eigenvectors (or more precisely, a basis of an invariant subspace) is needed to bound the convergence in a given set of iterative steps. Of course, even if $\|P_{Q_1}\|$ is constant, the convergence might still be dependent on the mesh parameter if $\|\bar{r}_j\|$ is heavily dependent on it.

We conclude this section illustrating in Figure 4.1 the bounds (4.1) and a bound for (4.2) for the GMRES convergence of a discretization of a two-dimensional model problem (2.2), with constants as in (2.3), preconditioned with (3.1), as described in section 3. Details on the choice of Q_1 are given later in the section 6.

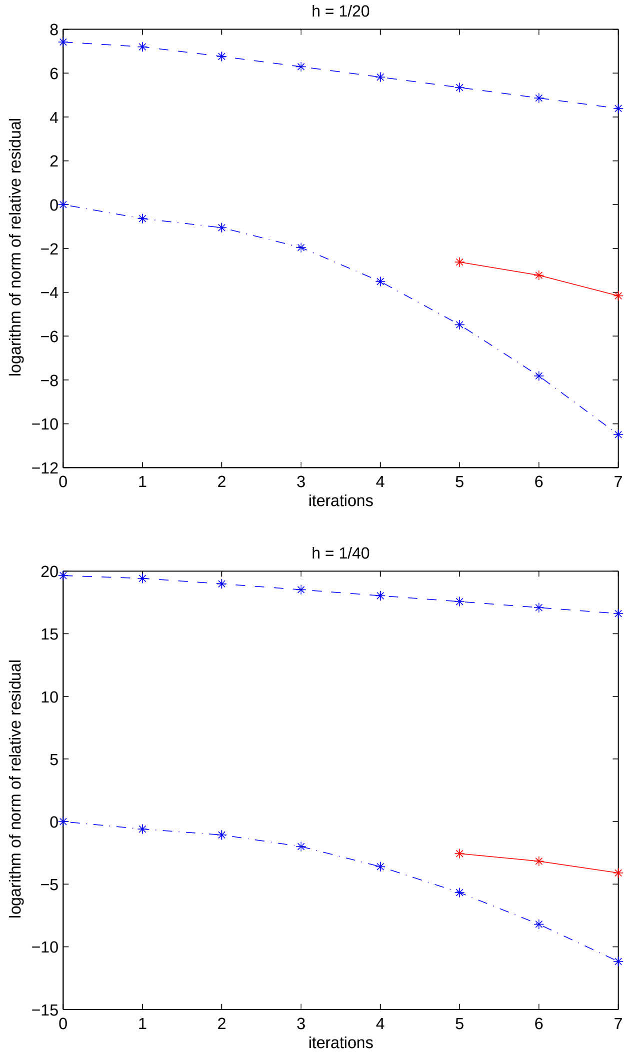

FIG. 4.1. *Preconditioned GMRES convergence and bounds for* (2.2), *with* (2.3). *Top: problem of size $N = 361$ ($h = 1/20$). Bottom: problem of size $N = 1521$ ($h = 1/40$). In both figures, the uppermost dashed line is the standard GMRES bound* (4.1), *the middle solid line is the a posteriori bound* (4.2) *and the lower dash-dotted line is the norm of the relative residual.*

5. Analysis and experiments for the one-dimensional model problem. For the model problem (2.1) preconditioned with additive Schwarz, Cai and Zou [1] consider two overlapping subdomains $\Omega_1 = [0, l_2]$ and $\Omega_2 = [l_1, 1]$, $l_1 < l_2$ fixed, and the mesh size $h = 1/(n + 1)$ varying with different values of n. They analyzed the preconditioned matrix $M^{-1}A$, and showed the following structure of its spectrum: For every node in the interior of regions without overlap, there is an eigenvalue $\lambda_1 = 1$, for every node in the interior of the overlap region, there is an eigenvalue $\lambda_2 = 2$, and the remaining two eigenvalues are $\lambda_3 = 1 - \sqrt{\frac{l_1(1-l_2)}{l_2(1-l_1)}}$ and $\lambda_4 = 1 + \sqrt{\frac{l_1(1-l_2)}{l_2(1-l_1)}}$. As mentioned in the introduction, they showed that the corresponding eigenvectors are such that $\kappa(Q)$ is of the order of $1/\sqrt{h}$.

We consider here exactly the same situation, and study how $\|P_{Q_1}\|$ depends on the interval length h. For our analysis we choose Q_1 to be the matrix with two fixed eigenvectors corresponding to $\lambda_2 = 2$. Let m_1 be the number of interior points of $\Omega_1 \setminus \Omega_2$, and let m_2 be the number of interior points of Ω_1. Then, the eigenvectors corresponding to $\lambda_2 = 2$ are the Euclidean vectors $e_{m_1+1}, \ldots, e_{m_2-1}$ [1]. Thus, we choose Q_1 to be a pair of these Euclidean vectors.

THEOREM 5.1. *Let $Ax = b$ be the linear system obtained from discretizing* (2.1), *let M^{-1} be the additive Schwarz preconditioner, let $Q = [Q_1, Q_0]$ be the eigenvector matrix of $M^{-1}A$ and $T = Q^{-*} = [T_1, T_0]$. Suppose that Q_1 is a matrix composed of two fixed eigenvectors corresponding to $\lambda_2 = 2$, and let $P_{Q_1} = Q_1 T_1^*$. Then it holds $\|P_{Q_1}\| \leq C$ for some small positive constant C of order 1.*

Proof. As it was shown in [1], $M^{-1}A$ has four different eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 1 - \sqrt{\frac{l_1(1-l_2)}{l_2(1-l_1)}}$ and $\lambda_4 = 1 + \sqrt{\frac{l_1(1-l_2)}{l_2(1-l_1)}}$. Let u and w be the eigenvectors corresponding to λ_3 and λ_4. Following [1], we can explicitely write $u = \alpha_3\sqrt{l_1 l_2}(\psi_{l_1}(x) + \psi_{l_2}(x))$, and $w = \alpha_4\sqrt{l_1 l_2}(\psi_{l_1}(x) - \psi_{l_2}(x))$, with $\psi_{l_1}(x)$ and $\psi_{l_2}(x)$ defined as follows:

$$\psi_{l_1}(x) = \begin{cases} 0 & x = 0 \\ \text{linear} & x \in (0, l_1) \\ 1 & x = l_1 \\ \text{linear} & x \in (l_1, l_2) \\ 0 & x \in [l_2, 1] \end{cases}, \quad \psi_{l_2(x)} = \begin{cases} 0 & x \in [0, l_1] \\ \text{linear} & x \in [l_1, l_2] \\ 1 & x = l_2 \\ \text{linear} & x \in (l_2, 1) \\ 0 & x = 1.0, \end{cases},$$

where α_3 and α_4 are chosen so that $\|u\|_2 = 1$ and $\|w\|_2 = 1$. Then $Q = [Q_1, Q_0] = I + UV$, where $U = (u - e_{m1}, w - e_{m2})$, and $V = (e_{m1}, e_{m2})^T$, with $m_1 = l_1/h$ and $m_2 = l_2/h$. We then have that $Q^{-1} = I - U(I + VU)^{-1}V = I - (z_{m1}e_{m1}^t + z_{m2}e_{m2}^t)/D$ where $D = u_{m1}w_{m2} - u_{m2}w_{m1}$, $z_{m1} = (w_{m2}(u - e_{m1}) - u_{m2}(w - e_{m2}))$, and $z_{m2} = (-w_{m1}(u - e_{m1}) + u_{m1}(w - e_{m2}))$.

If we choose $Q_1 = [e_{m_1+i}, e_{m_2-j}]$, where both i and j are less than $m_2 - m_1$, it is easy to compute

$$T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \ldots & \ldots \\ \eta(1 - \frac{ih}{l_2-l_1}) & \frac{-\eta jh}{l_2-l_1} \\ 0 & 0 \\ \ldots & \ldots \\ 1 & 0 \\ 0 & 0 \\ \ldots & \ldots \\ 0 & 1 \\ \ldots & \ldots \\ 0 & 0 \\ \frac{-\eta ih}{l_2-l_1} & \eta(1 - \frac{jh}{l_2-l_1}) \\ 0 & 0 \\ \ldots & \ldots \\ 0 & 0 \end{bmatrix},$$

where $\eta = 2\alpha_3\alpha_4 l_1 l_2 < 2$. Therefore, $P_{Q_1} = Q_1 T_1^*$ is an $n \times n$ matrix with only six nonzero elements. A direct calculation gives

$$\begin{aligned} 1 &\leq \|P_{Q_1}\|_2 \leq \|P_{Q_1}\|_F = \\ &= \sqrt{2 + \eta^2((1 - ih/(l_2 - l_1))^2 + (1 - jh/(l_2 - l_1))^2 + (ih/(l_2 - l_1))^2 + (jh/(l_2 - l_1))^2)} \\ &\leq \sqrt{2} + \eta(ih/l_2 - l_1) + \eta(1 - ih/l_2 - l_1) + \eta(jh/l_2 - l_1) + \eta(1 - jh/l_2 - l_1) \\ &= \sqrt{2} + 2\eta \leq 5.5. \qquad \square \end{aligned}$$

We illustrate the result of this theorem by computing $\|P_{Q_1}\|$ for eight different values of h, namely $h = 1/2^k$, for $k = 3, 4, \ldots, 10$,with fixed values of $l_1 = 1/4$ and $l_2 = 3/4$ (other values of the overlap give similar results). We show these values in Figure 5.1. Note the narrow range of the vertical axis.

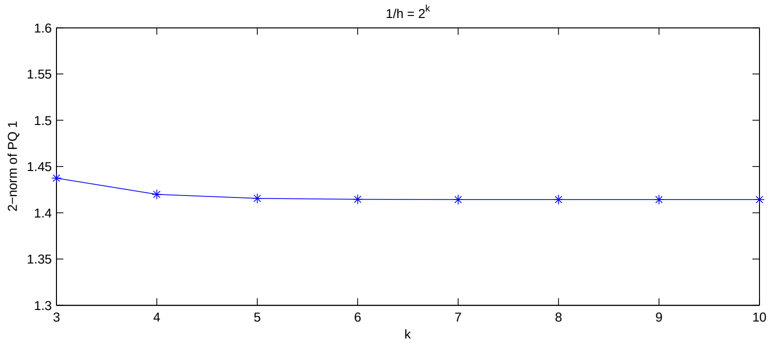

FIG. 5.1. *Norm of P_{Q_1} for various values of h for the one-dimensional problem* (2.1)

We end this section by showing, in Figure 5.2, the convergence curves for the same problem used in Figure 5.1, i.e., for the additive Schwarz preconditioned GMRES for the problem (2.1). The reader can appreciate how the convergence curves behave in almost identical manner, i.e., independent of the mesh size.

6. Numerical experiments for two-dimensional model problems. In this section we present numerical experiments for different mesh size h for the two-dimensional model problems (2.2). We have $h = 1/(n+1)$ for $n = 9, 19, 39, 59$, i.e., the problem size is $n^2 = 81, 361, 1521, 3481$. We report for each of the three cases (2.3)-(2.5) tables with the number of iterations needed to reduce the relative residual norm to 10^{-8}, as well as the value of $\|P_{Q_1}\|$. In most cases, as in the one-dimensional problem already discussed, $\lambda = 2$ is a multiple eigenvalue, and we choose two eigenvectors of $M^{-1}A$ corresponding to this eigenvalue. In a few other cases, we chose some other pair of eigenvectors. We consider four different decompositions of the unit square:

1. Two overlapping strips, where $\Omega_1 = [0, 0.6] \times [0, 1]$, and $\Omega_2 = [0.4, 1] \times [0, 1]$.

2. Three overlapping strips, where $\Omega_1 = [0, 0.3] \times [0, 1]$, $\Omega_2 = [0.2, 0.5] \times [0, 1]$, and $\Omega_3 = [0.4, 1] \times [0, 1]$.

3. Four overlapping strips, where $\Omega_1 = [0, 0.3] \times [0, 1]$, $\Omega_2 = [0.2, 0.5] \times [0, 1]$, $\Omega_3 = [0.4, 0.7] \times [0, 1]$, and $\Omega_4 = [0.6, 1] \times [0, 1]$.

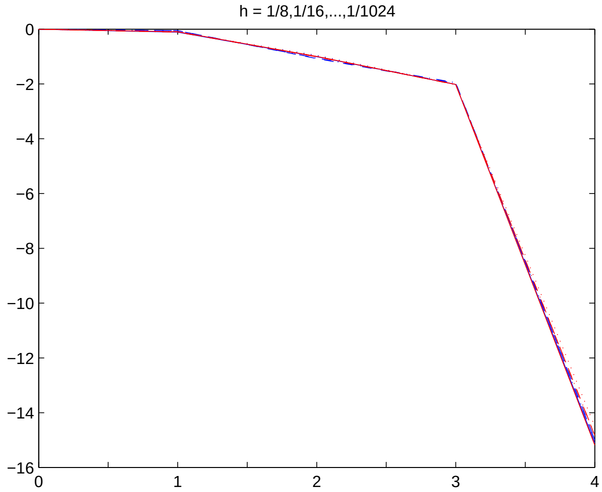

FIG. 5.2. *Preconditioned GMRES convergence for various values of h for the one-dimensional model problem* (2.1)

4. Four overlapping squares, where $\Omega_1 = [0, 0.6] \times [0, 0.6]$, $\Omega_2 = [0.4, 1] \times [0, 0.6]$, $\Omega_3 = [0, 0.6] \times [0.4, 1]$, and $\Omega_4 = [0.4, 1] \times [0.4, 1]$

We consider two different right hand sides, b_1 and b_2, discretizations of some function f such that the corresponding exact solution of (2.2) are the functions

$$u = \sin \pi x \sin \pi y, \tag{6.1}$$

$$\text{or} \quad u = e^{x+y} \sin(2\pi x) \sin(2\pi y), \tag{6.2}$$

respectively.

We begin with the simplest model problem, namely (2.2)-(2.3), right hand side b_1 corresponding to (6.1), and two strips. We display in Figure 6.1 GMRES the convergence curves for the four discretizations, where dash-dotted line is for $n = 9$, dashed line for $n = 19$, solid line for $n = 39$, and dotted line for $n = 59$. We present in Table 6.1 the corresponding values of $\|P_{Q_1}\|$.

$1/h$	10	20	40	60
order of the matrix ($N = n^2$)	81	361	1521	3481
iterations	7	7	7	7
$\|P_{Q_1}\|_2$	1.2489	1.0675	1.1897	1.0482

TABLE 6.1
Norm of P_{Q_1} and number of iterations for the two-dimensional problem (2.2)-(2.3)*, two strips.*

We consider three other problems of the form (2.2). In tables 6.2–6.4 we report convergence information and $\|P_{Q_1}\|$ for the two-dimensional problem (2.2) with $a = 1$, $b = 0$; $a = 5$, $b = -10$; and $a = 10$, $b = 20$; respectively. In Table 6.2, the right hand side use is b_1 corresponding to (6.1), while in the latter two, we report results using both right hand sides. It can be observed in all our tables that for each of the model problems studied here, both the number of iterations and $\|P_{Q_1}\|$ are pretty constant

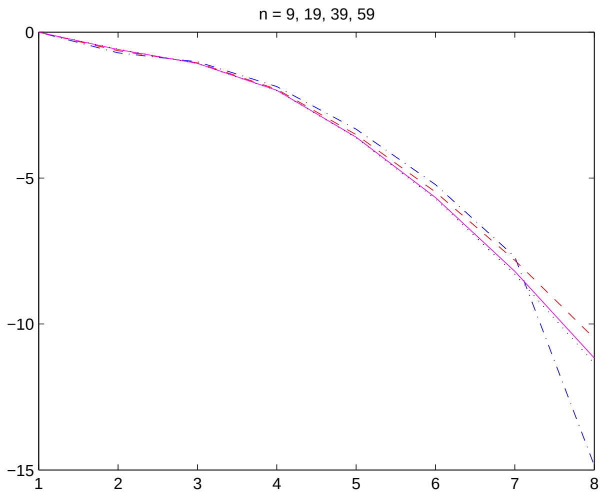

FIG. 6.1. *Preconditioned GMRES convergence for the two-dimensional model problem* (2.2)-(2.3)

while varying the mesh size h. If we denote by H the width of a subdomain, we also observe that there is a slight increase in the number of iteration as H decreases (for a fixed h). This is consistent with the theory that says that the number of iterations may grow at a rate of the order of $1/H^2$; see [12, p. 17], [14].

	$1/h$	10	20	40	60
	$N = n^2$	81	361	1521	3481
Two strips	iterations	9	9	8	9
	$\|P_{Q_1}\|_2$	1.1909	1.1566	1.1892	1.2689
Three strips	iterations	11	9	10	11
	$\|P_{Q_1}\|_2$	1.4404	1.2215	1.3624	1.1093
Four strips	iterations	13	11	13	15
	$\|P_{Q_1}\|_2$	1.1346	1.2455	1.2339	1.2278
Four squares	iterations	11	12	12	14
	$\|P_{Q_1}\|_2$	1.0696	1.1863	1.1338	1.1719

TABLE 6.2

Norm of P_{Q_1} *and number of iterations for the problem* (2.2)*, with* $a = 1$*,* $b = 0$*.*

We illustrate the convergence behavior of the additive Schwarz preconditioned GMRES in Figures 6.2 and 6.3, where we report the convergence curves for the problem (2.2) for the right hand side corresponding to (6.1), with $a = 5$, $b = -10$, and $a = 10$, $b = 20$, respectively. In each figure we show the convergence for for two strips (left), and four squares (right), for four discretizations. It can be appreciated that the convergence curves are very similar, for all values of the discretization parameter h.

We mention that the one-dimensional model problem (2.1), and the two-dimensional model problem (2.2) with parameters as in (2.3), (2.4), have real spectrum, while (2.2) with (2.5) does not. We present in Figure 6.4 the spectra of the matrix A corresponding to the latter problem with $n = 39$, and parameter values $a = 5$, $b = -10$ (left) and $a = 10$, $b = 20$ (right). In Figures 6.5 and 6.6 we show

	$1/h$ $N = n^2$	10 81	20 361	40 1521	60 3481
Two strips	iterations for (6.1)	9	8	7	7
	iterations for (6.2)	10	10	9	9
	$\|P_{Q_1}\|_2$	1.2166	1.2705	1.2849	1.4816
Three strips	iterations for (6.1)	10	7	9	8
	iterations for (6.2)	12	9	10	10
	$\|P_{Q_1}\|_2$	3.4438	2.0393	2.0337	1.9882
Four strips	iterations for (6.1)	12	10	13	14
	iterations for (6.2)	14	12	14	15
	$\|P_{Q_1}\|_2$	7.2526	1.9793	1.7738	2.0299
Four squares	iterations for (6.1)	12	12	13	13
	iterations for (6.2)	12	14	14	15
	$\|P_{Q_1}\|_2$	1.4839	1.2394	1.2277	1.2351

TABLE 6.3
Norm of P_{Q_1} and number of iterations for the problem (2.2), *with* $a = 5$, $b = -10$.

	$1/h$ $N = n^2$	10 81	20 361	40 1521	60 3481
Two strips	iterations for (6.1)	7	7	7	7
	iterations for (6.2)	8	8	8	8
	$\|P_{Q_1}\|_2$	1.3334	1.6662	1.4529	1.9644
Three strips	iterations for (6.1)	10	6	7	7
	iterations for (6.2)	11	8	8	8
	$\|P_{Q_1}\|_2$	1.1038	1.5397	1.1176	1.0519
Four strips	iterations for (6.1)	12	9	10	10
	iterations for (6.2)	12	10	11	11
	$\|P_{Q_1}\|_2$	1.1174	1.5364	0.9975	1.0006
Four squares	iterations for (6.1)	11	12	12	12
	iterations for (6.2)	12	12	13	13
	$\|P_{Q_1}\|_2$	2.0682	1.6720	1.6641	1.8065

TABLE 6.4
Norm of P_{Q_1} and number of iterations for the problem (2.2), *with* $a = 10$, $b = 20$.

the spectra of the preconditioned systems $M^{-1}A$ in the case of four strips (left) and four squares (right). The clustering effect of the additive Schwarz preconditioning for these problems can be observed.

7. Conclusion and discussion. It is well-known that the bound (4.1) can be very pessimistic. Therefore any non-optimality analysis based on quantities in this bound may not fully characterize the convergence behavior of the methods. We make the case that a quantity used in the a posteriori bound (4.2) may reflect better the behavior of the iterative method (as the iterations progress) than the condition number of the eigenvector matrix. For the model problems presented in this paper, we have shown that additive Schwarz preconditioned GMRES without coarse grid correction is either optimal or close to optimal, i.e, its convergence is pretty independent of the finite element mesh size.

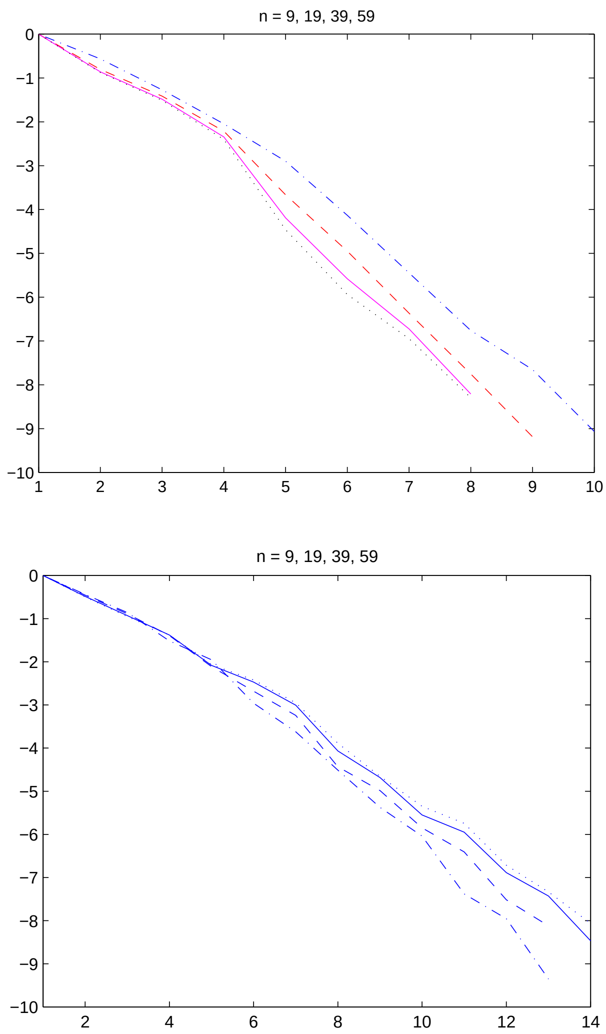

Fig. 6.2. *Preconditioned GMRES convergence for the problem* (2.2)*, with* $a = 5$*,* $b = -10$*. Top: two strips. Bottom: four squares. Dash-dotted:* $n = 9$*. Dashed:* $n = 19$*. Solid:* $n = 39$*. Dotted:* $n = 59$*.*

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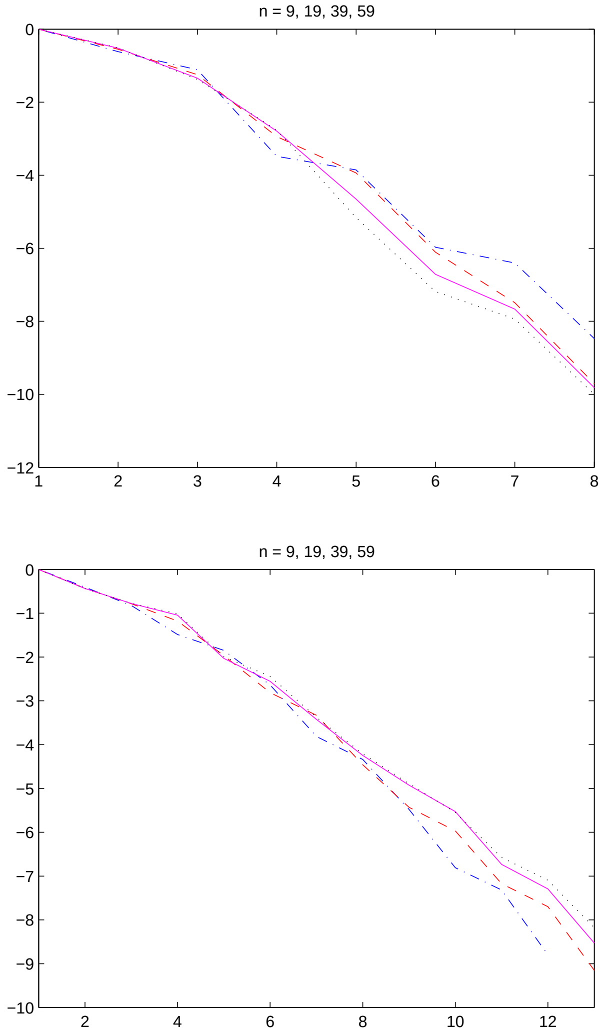

FIG. 6.3. *Preconditioned GMRES convergence for the problem* (2.2)*, with* $a = 10$*,* $b = 20$*. Top: two strips. Bottom: four squares. Dash-dotted:* $n = 9$*. Dashed:* $n = 19$*. Solid:* $n = 39$*. Dotted:* $n = 59$.

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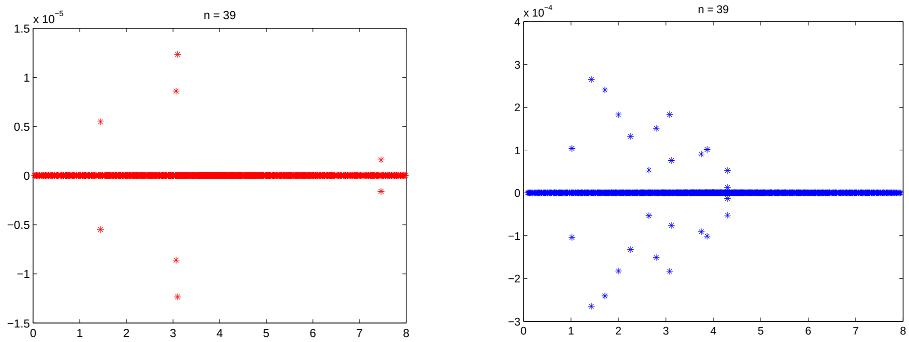

FIG. 6.4. *Spectra of the discretized problem* (2.2) *($n = 39$) with $a = 5$, $b = -10$ (left, with vertical scale $\times 10^{-5}$) and with $a = 10$, $b = 20$ (right, with vertical scale $\times 10^{-4}$).*

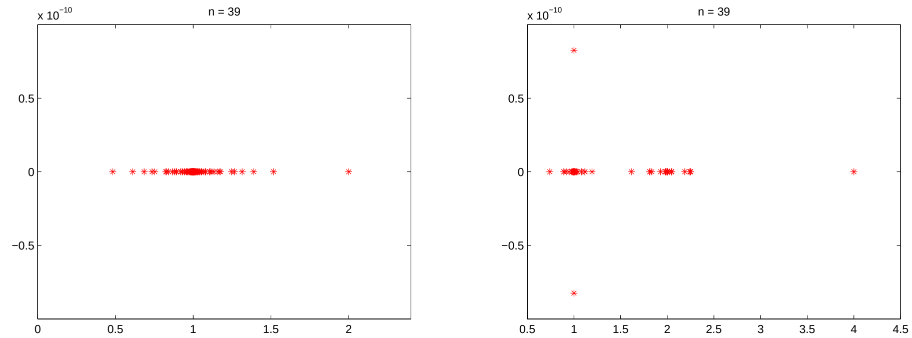

FIG. 6.5. *Spectra of the discretized problem* (2.2) *($n = 39$) with $a = 5$, $b = -10$, preconditioned with additive Schwarz. Domain using four strips (left) and four squares (right). Vertical scale is $\times 10^{-10}$.*

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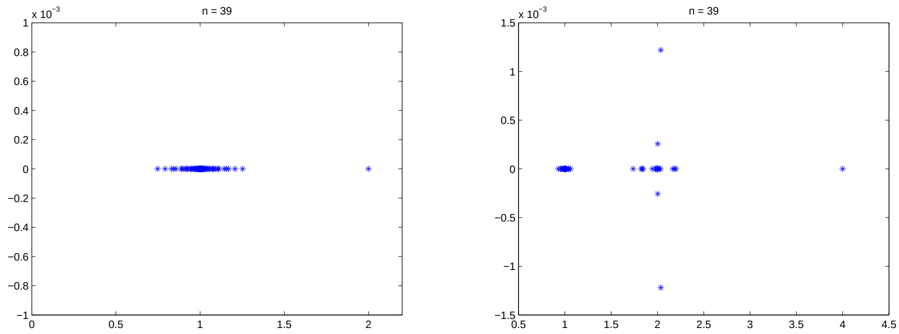

FIG. 6.6. *Spectra of the discretized problem* (2.2) *($n = 39$) with $a = 10$, $b = 20$, preconditioned with additive Schwarz. Domain using four strips (left) and four squares (right). Vertical scale is* $\times 10^{-3}$.

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