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Conditions for strict inequality in comparisons of spectral radii of splittings of different matrices

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Abstract

We present new comparison theorems for the spectral radii of matrices arising from splittings of different matrices under nonnegativity assumptions. Our focus is on establishing strict inequalities of the spectral radii without imposing strict inequalities of the matrices, but we also obtain new results for nonstrict inequalities of the spectral radii. We emphasize two different approaches, one combinatorial and the other analytic and discuss their merits in the light of the results obtained. We try to get fairly general results and indicate by counter-examples that some of our hypotheses cannot be relaxed in certain directions.

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1. Introduction

We study comparison theorems between nonnegative splittings of two different matrices. Our focus is on strict inequalities for the spectral radii, while the inequalities in the hypotheses (with respect to the nonnegative cone of matrices) are not strict.

We present two fundamentally different approaches to this topic. The first one is combinatorial in nature and makes explicit use of the graph of the matrix involved. The second approach is analytic in nature and relies on topological and algebraic arguments. Both these approaches are interesting by their methodology. They usually complement each other. For example, in [13,14] graph theoretical arguments were used to prove certain results, while in [9] analytical arguments were used for the same results. In our comparison theorems here it turns out that the graph theoretical approach is somehow restricted to considering M-matrices, whereas the analytical approach allows us to obtain further results involving general monotone matrices, i.e., matrices with nonnegative inverses.

Comparison theorems between the spectral radii of matrices are a useful tool in the analysis of the rate of convergence of iterative methods or for judging the efficiency of preconditioners. There is also a connection to population dynamics; see, e.g., [7] and the references given therein.

The paper is organized as follows. In Section 2 we derive a strict inequality result using the combinatorial approach. We give two different versions of this result. The first applies to iterative methods obtained through matrix splittings, whereas the second is more appropriate, e.g., when studying population dynamics. In Section 3 we then present several generalizations of this result in the splitting formulation using an analytic approach. There, we introduce and use various notions of nonnegative splittings. Further extensions are presented in Section 4. In particular, one can appreciate the variety of comparison results that can be obtained for splittings of different matrices. This section also contains a new result for nonstrict inequalities.

2. The combinatorial approach

We use the notation $A \geqslant A'$ for two real matrices of equal size if each entry of the difference $A - A'$ is nonnegative. We write $A > A'$ if each entry of the difference is positive. A matrix $A \geqslant O$ $(A > O)$ is called nonnegative (positive). We will often consider relations $A \geqslant A'$ with equality excluded $(A \neq A')$ for which we write $A \gneq A'$.

A nonsingular M-matrix A is such that it can be written as $A = \sigma I - T$ with $T \geqslant O$ and $\sigma > \rho(T)$; see, e.g., [2,15]. Alternatively, A is a nonsingular M-matrix if it can be expressed as $A = \sigma(I - T')$ with $\sigma > 0$, $T' \geqslant O$ and $\rho(T') < 1$.

Given a square matrix A, $A = M - N$ is called a splitting if M is nonsingular. A splitting is *regular* if $M^{-1} \geqslant O$ and $N \geqslant O$ [15]. It is an *M-splitting* if M is an M-matrix and $N \geqslant O$ [13]; see further Definition 3.3.

We can state now our main theorem for splittings of M-matrices.

Theorem 2.1. *Let $A_1 = M_1 - N$ and $A_2 = M_2 - N$ be M-splittings of A_1 and A_2, respectively, where*

$$M_1 \lneq M_2 \quad \text{and} \quad N \gneq O. \tag{1}$$

If $A_1^{-1} > O$, then

$$0 < \rho(NM_2^{-1}) < \rho(NM_1^{-1}). \tag{2}$$

The importance of this theorem is that (2) is strict while hypotheses (1) are not. As we shall see, Theorem 2.1 can be considered a reformulation of part (2) of the following result.

Theorem 2.2. (i) *Let T and F be square nonnegative matrices. Suppose that the spectral radius satisfies $\rho(T) < 1$. Assume also that $F \neq O$ and that $F + T$ is irreducible. Let $Q = F(I - T)^{-1}$. Then, after a permutation similarity,*

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ O & O \end{pmatrix},$$

where Q_{11} is a nontrivial irreducible nonnegative matrix, Q_{12} is a nonnegative matrix every column of which has a positive entry, and the zero rows of Q correspond to the zero rows of F, if any. Further $\rho(Q) > 0$.

(ii) *In addition, let $O \leqslant T' \lneq T$, and put $Q' = F(I - T')^{-1}$. Then, after the same permutation similarity and partitioning,*

$$Q' = \begin{pmatrix} Q'_{11} & Q'_{12} \\ O & O \end{pmatrix},$$

where $Q'_{11} \lneq Q_{11}$ and $\rho(Q') < \rho(Q)$.

Again, we observe that the point of the theorem is that the inequalities on the spectral radii in Parts (i) and (ii) are *strict*, since it is trivial that $O \leqslant Q' \leqslant Q$ and hence it follows by standard Perron–Frobenius theory that $0 \leqslant \rho(Q') \leqslant \rho(Q)$.

We will present a complete graph theoretic proof of Theorem 2.2, although Part (i) was presented, in terms of the M-splitting of Theorem 2.1, at the Linear Algebra meeting Oberwolfach in 1982 and appeared as Lemma 3.4 of [13]; see also [14,17].

Before starting the proof, let us introduce standard terminology for graphs of nonnegative matrices; see, e.g., [13]. Let n be a positive integer. Then a *path* (without further qualification) will be a sequence $\mathfrak{p} = (i(0), \ldots, i(s))$ of positive integers $i(r)$, $1 \leqslant i(r) \leqslant n$, $r = 0, \ldots, s$. A path is usually called an *arc* if $s = 1$. Let A be a nonnegative $n \times n$ matrix. We call the path $\mathfrak{p}$ an *A-path* or a *path in A* if $a_{i(r-1),i(r)} > 0$, $r = 1, \ldots, s$, and we use similar terminology for arcs.

For any path $\mathfrak{p}$, the *path product* $\mathfrak{p}(A)$ is defined by

$$\mathfrak{p}(A) = a_{i(0),i(1)} \cdots a_{i(s-1),i(s)}.$$

Let F and T be two nonnegative $n \times n$ matrices and suppose the spectral radius of T satisfies $\rho(T) < 1$. We are here concerned with the matrix $Q = F(I - T)^{-1}$, as essentially was [13] where Q^{T} was considered. Since $Q = F(I + T + T^2 + \cdots)$, the elements of Q are easily computed to be

$$q_{i,j} = \sum_{\mathfrak{a},\mathfrak{p}} \mathfrak{a}(F)\mathfrak{p}(T), \tag{3}$$

where the summation is taken over all arcs $\mathfrak{a} = (i, k)$ and all paths $\mathfrak{p}$ from k to j *or*—considering only nonzero summands—over all F-arcs $\mathfrak{a} = (i, k)$ and all T-paths $\mathfrak{p}$ from k to j, see [13, Theorem 2.7]. In view of this, we call a path *relevant* if its first arc is an F-arc and all other arcs (if any) are T-arcs. By (3), $q_{i,j} > 0$ if and only if there is a relevant path from i to j. If $\mathfrak{q} = (\mathfrak{a}, \mathfrak{p})$ is a relevant path with first arc $\mathfrak{a}$, we define $\mathfrak{q}(F, T) = \mathfrak{a}(F)\mathfrak{p}(T)$. Hence (3) may be rewritten as

$$q_{i,j} = \sum_{\mathfrak{q}\in\Re(i,j)} \mathfrak{q}(F, T), \tag{4}$$

where $\Re(i, j)$ is the set of all relevant paths from i to j.

Our proofs depend on the following graph theoretic remark:

Remark 2.3. Let $\mathfrak{p}$ be an $(F + T)$-path that begins with an F-arc. Then $\mathfrak{p}$ may be decomposed as $\mathfrak{p} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$, where each $\mathfrak{q}_\ell$, $\ell = 1, \ldots, s$, is a relevant path. As an additional condition, we may even impose that each $\mathfrak{q}_\ell$, $\ell = 2, \ldots, s$, begins with an F-arc which is *not* a T-arc. Since each $\mathfrak{q}_\ell$ begins with an F-arc which corresponds to a positive element of Q, it follows that, if $\mathfrak{p}$ is an $(F + T)$-path from i to j that begins with an F-arc, there is a Q-path from i to j (and the converse also holds).

Proof of Theorem 2.2. Let $\mathscr{B}$ be the set of vertices which are the starts of F-arcs and let $\mathscr{R}$ be the complement of $\mathscr{B}$ in $\{1, \ldots, n\}$. Apply a permutation similarity to all matrices so that $\mathscr{B} = \{1, \ldots, k\}$, $1 \leqslant k \leqslant n$. Note that $\mathscr{B}$ is nonempty and partition Q so that Q_{11} is $k \times k$.

Let b be any vertex of $\mathscr{B}$ and let d be any vertex. Let $\mathfrak{a} = (b, c)$ be an F-arc. Since $F + T$ is irreducible, there exists an $(F + T)$-path $\mathfrak{r}$ from c to d and let $\mathfrak{p} = (\mathfrak{a}, \mathfrak{r})$. Suppose that $\mathfrak{p} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$ is a decomposition into relevant paths. In view of Remark 2.3 it follows that there is a Q-path from b to d. Hence there is a nonempty path in Q from every vertex of $\mathscr{B}$ to every other vertex. It follows that every column of $[Q_{11} \;\; Q_{12}]$ contains a positive element. Now let $d \in \mathscr{B}$. Then every $\mathfrak{q}_\ell$, $\ell = 1, ..., s$, corresponds to a relevant path beginning and ending at an element of Q_{11}. Hence there exists a Q_{11}-path from b to d and it follows by a standard result that Q_{11} is irreducible. Also, there is no relevant path from a vertex of $\mathscr{R}$ since there is no F-arc that starts there. This proves that $Q_{21} = O$ and $Q_{22} = O$. Since Q_{11} is a nontrivial irreducible matrix (viz. $Q_{11} \neq O$), we have $0 < \rho(Q_{11}) = \rho(Q)$. This proves Part (i).

Since $O \leqslant T' \lneq T$, there exists an arc (i, j) of T such that $0 \leqslant t'_{i,j} < t_{i,j}$. Let b be any vertex of $\mathscr{B}$ and let (b, c) be an F-arc. Since $F + T$ is irreducible, there exists an $(F + T)$-path $\mathfrak{p}$ which begins with the arc (b, c), then continues through the arc (i, j) (repeating (b, c) if it happens that $(b, c) = (i, j)$) and continues back to b. We decompose the path $\mathfrak{p} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$ into relevant paths that satisfy the additional condition that each $\mathfrak{q}_\ell$, $\ell > 1$, begins with an F-arc which is not a T-arc. Then there is an ℓ, $1 \leqslant \ell \leqslant s$, such that (i, j) is a noninitial arc of $\mathfrak{q}_\ell$ and $\mathfrak{q}_\ell(F, T') < \mathfrak{q}_\ell(F, T)$, since $t'_{i,j} < t_{ij}$. The relevant path $\mathfrak{q}_\ell$ begins and ends at vertices of $\mathscr{B}$; say it is an (a, d) path. Since for all relevant paths $\mathfrak{q}$ we have $\mathfrak{q}(F, T') \leqslant \mathfrak{q}(F, T)$, it follows by (4) that $q'_{a,d} < q_{a,d}$. This proves that $Q'_{11} \lneq Q_{11}$ and, since Q_{11} is irreducible, it follows that $\rho(Q') = \rho(Q'_{11}) < \rho(Q_{11}) = \rho(Q)$, and Part (ii) is proved. □

Remark 2.4. Above we have essentially proved one direction of the following result: $q'_{a,d} < q_{a,d}$ if and only if there exists a relevant path from a to d which contains a noninitial arc (i, j) such that $t'_{i,j} < t_{i,j}$.

As a corollary of Part (ii) of Theorem 2.2 we have the following result first proved as [7, Theorem 4.4].

Proposition 2.5. *Let T and F be square nonnegative matrices with $F \neq O$ and $F + T$ irreducible. Suppose that the spectral radius $\rho(T)$ of T satisfies $\rho(T) < 1$. Then $\rho(F(I - T)^{-1}) > 0$. Moreover, let $s > \rho(T)$. Then*

$$\rho(F(I-T)^{-1}) < \rho\left(F\left(I - \frac{1}{s}T\right)^{-1}\right) \quad \text{if } s < 1$$

and

$$\rho(F(I-T)^{-1}) > \rho\left(F\left(I - \frac{1}{s}T\right)^{-1}\right) \quad \text{if } s > 1.$$

The above proposition has a direct interpretation in models from population dynamics where T and F represent the transition and fertility matrix, respectively, and s is the growth rate. We refer to [7] for details.

We now turn to prove Theorem 2.1 using Theorem 2.2.

Proof of Theorem 2.1. Since M_i, $i = 1, 2$, is an M-matrix, we may write $M_i = c_i(I - \widetilde{T}_i)$, where $c_i > 0$, $\widetilde{T}_i \geq O$, $\rho(\widetilde{T}_i) < 1$ for $i = 1, 2$. Let $c = \max\{c_1, c_2\}$ and write

$$M_i = c_i(I - \widetilde{T}_i) = c\left(I - \left(\frac{c - c_i}{c}I + \frac{c_i}{c}\widetilde{T}_i\right)\right) = c\,(I - T_i)\,.$$

Here,

$$T_i = \frac{c - c_i}{c}I + \frac{c_i}{c}\widetilde{T}_i \geqslant O \quad \text{and} \quad \rho(T_i) = \frac{c - c_i}{c} + \frac{c_i}{c}\rho(\widetilde{T}_i) < 1.$$

Since $O < A_1^{-1} = c^{-1}(I - N - T_1)^{-1}$, it follows that $N + T_1$ is irreducible. Thus the theorem is implied by Theorem 2.2 (where $F = N$, $T = T_1$, $T' = T_2$). □

The proof just presented shows that Theorem 2.2 implies Theorem 2.1. If we disregard the statement on the structure of Q and Q' in Theorem 2.2, the converse also holds: Theorem 2.1 implies Theorem 2.2. This follows indeed easily, since in Theorem 2.2 we may without loss of generality scale $F = N$ by a positive constant so that $\rho(F + T) < 1$, which yields $(I - F - T)^{-1} > O$.

We end this section with a couple of examples showing that the hypotheses $M_1 \lneqq M_2$ and $A_1^{-1} > O$ in Theorem 2.1 cannot be weakened.

First observe that for M-matrices M_1, M_2, $M_1 \lneqq M_2$ implies that ${M_1}^{-1} \gneqq {M_2}^{-1}$ but not conversely. It is tempting to assume that Theorem 2.1 remains valid if the hypothesis $M_1 \lneqq M_2$ is replaced by ${M_1}^{-1} \gneqq {M_2}^{-1}$, but this is not so as is easily shown by the following example.

Example 2.6. Let

$$M_1^{-1} = \begin{pmatrix} 1 & .5 \\ .5 & 1 \end{pmatrix}, \quad M_2^{-1} = \begin{pmatrix} 1 & .5 \\ .4 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} .1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then ${M_1}^{-1} \gneqq {M_2}^{-1}$ and $(M_1 - N)^{-1} > O$ but $N{M_1}^{-1} = N{M_2}^{-1}$.

Remark 2.7. If we assume *strict* inequality ${M_1}^{-1} > {M_2}^{-1}$, the other assumptions of Theorem 2.1 remaining unchanged, then the conclusion $\rho(NM_2^{-1}) < \rho(NM_1^{-1})$ still holds. This can be seen by considering the reducible normal form; see, e.g., [15]. Observe that $O \leqslant NM_2^{-1} \lneqq NM_1^{-1}$ follows immediately from the assumptions, and NM_1^{-1} has at least one positive diagonal element. So the reducible normal form of NM_1^{-1} has at least one nontrivial (irreducible) diagonal block. But all positive elements in each such block are strictly decreased when passing from NM_1^{-1} to NM_2^{-1}, so the spectral radius of each block strictly decreases. This gives $\rho(NM_1^{-1}) > \rho(NM_2^{-1})$ by standard Perron–Frobenius theory.

It is also easy to see that we cannot omit the condition that $A_1^{-1} > O$ from Theorem 2.1 (or, equivalently, that $F + T$ is irreducible in Theorem 2.2).

Example 2.8. Let

$$N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = I.$$

Then

$$NM_1^{-1} = NM_2^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. An analytic approach

The purpose of this section is to develop generalizations for Theorem 2.1. We will be able to dispense with the M-splitting hypothesis by assuming more general nonnegativity hypotheses. Section 4 will contain further generalizations together with a discussion of possible applications; see Remark 4.7. Our approach is now analytical rather than combinatorial as exemplified by the following auxiliary result.

Lemma 3.1. *Let B_0 and B_1 be two square nonsingular matrices such that $B_0 \lneqq B_1$ and $B_0^{-1} > O$ as well as $B_1^{-1} \geqslant O$. Then $B_0^{-1} > B_1^{-1}$.*

Proof. Define the family of matrices $B_\alpha = (1-\alpha)B_0 + \alpha B_1$, $\alpha \in [0, 1]$. Clearly, we have $B_\alpha \lneqq B_\beta$ for $\alpha < \beta$. If B_α, B_β are nonsingular, we also have

$$B_\alpha^{-1} - B_\beta^{-1} = B_\beta^{-1}\left(B_\beta - B_\alpha\right)B_\alpha^{-1}. \tag{5}$$

By continuity, there exists a small positive $\gamma \in (0, 1)$ such that B_γ is nonsingular with $B_\gamma^{-1} > O$. Since $B_\gamma - B_0$ is nonnegative with at least one positive entry, the product $B_\gamma^{-1}(B_\gamma - B_0)$ has at least one positive column. Since $B_0^{-1} > O$, the product $B_\gamma^{-1}(B_\gamma - B_0)B_0^{-1}$ is therefore positive. From (5) (with $\alpha = 0$, $\beta = \gamma$) we therefore get

$$B_0^{-1} > B_\gamma^{-1}. \tag{6}$$

We also have $B_1 - B_\gamma \geqslant O$, and since $B_1^{-1} \geqslant O$, $B_\gamma^{-1} > O$ we have

$$B_\gamma^{-1} \geqslant B_1^{-1}, \tag{7}$$

again from (5) (with $\alpha = \gamma$, $\beta = 1$). Combining (6) and (7) we obtain the desired strict inequality $B_0^{-1} > B_1^{-1}$. $\square$

We also need the following, fairly standard auxiliary result; see, e.g., [8]. For the sake of completeness, we reproduce a proof here. Note that T need not be irreducible.

Lemma 3.2. *Let T be a nonnegative matrix, let x be a nonnegative nonzero vector and α a positive scalar.*

(i) *If $Tx \geqslant \alpha x$, then $\rho(T) \geqslant \alpha$. Moreover, if $Tx > \alpha x$, then $\rho(T) > \alpha$.*
(ii) *If $Tx < \alpha x$, then $\rho(T) < \alpha$.*

Proof. To show (i), assume that $Tx \geqslant \alpha x$ but $\rho(T) < \alpha$. Then

$$\left(I - \frac{1}{\alpha}T\right)^{-1} = \sum_{\nu=0}^{\infty}\left(\frac{1}{\alpha}T\right)^{\nu} \geqslant O.$$

Therefore,

$$0 \leqslant \left(I - \frac{1}{\alpha}T\right)^{-1}(Tx - \alpha x) = -\alpha x.$$

Since we had $x \geqslant 0$ this implies $x = 0$, a contradiction. This proves Part (i) for the nonstrict inequalities. If $Tx > \alpha x$ we can choose $\alpha_1 > \alpha$ for which still $Tx \geqslant \alpha_1 x$. By what we have already proved we get $\rho(T) \geqslant \alpha_1$ which implies $\rho(T) > \alpha$.

To show (ii), note that $x > 0$. The matrix $\frac{1}{\alpha}T$ then satisfies $\frac{1}{\alpha}Tx < \beta x$ with $0 < \beta < 1$. This implies

$$0 \leqslant \left(\frac{1}{\alpha}T\right)^n x < \beta^n x, \quad n = 0, 1, \ldots$$

This shows that $(\frac{1}{\alpha}T)^n$ tends to zero as every entry tends to zero, so that $\rho(T) < \alpha$. □

In order to formulate and appreciate the generalization of Theorem 2.1 let us recall the following—now near-standard—terminology; see, e.g., [3,16].

Definition 3.3. A splitting $A = M - N$ is called:

(i) *weak nonnegative of first type* if $M^{-1} \geqslant O$ and $M^{-1}N \geqslant O$,
(ii) *weak nonnegative of second type* if $M^{-1} \geqslant O$ and $NM^{-1} \geqslant O$.
(iii) *nonnegative* if it is weak nonnegative of both types.

Clearly, a regular splitting (and thus an M-splitting) is also weak nonnegative, and a weak nonnegative splitting is weak nonnegative of either type.

The following known results on weak nonnegative splittings are very important for our investigations. As it is usually done, we state these results in terms of the spectral radius of the "iteration matrix" $M^{-1}N$. But note that we could as well take NM^{-1} (as we did in Theorem 2.1) since $\rho(M^{-1}N) = \rho(NM^{-1})$.

Theorem 3.4. *Let A be a nonsingular matrix with* $A^{-1} \geqslant O$.

(i) *Let* $A = M - N$ *be a weak nonnegative splitting of either type. Then* $\rho(M^{-1}N) < 1$.
(ii) *If the splitting* $A = M - N$ *is weak nonnegative of second type, there exists a vector* $x \gneqq 0$ *such that* $M^{-1}Nx = \rho(M^{-1}N)x$ *and* $Ax \geqslant 0$ *as well as* $Nx \gneqq 0$.

Proof. For the first type, Part (i) goes back to [11], whereas for second type splittings it was given in [16]. The major Part of (ii) was proved in [1, Lemma 2.8] except for the inequality $Nx \gneqq 0$. To prove this, let us write $\rho = \rho(M^{-1}N)$. Note that we have $Nx = \rho Mx$ which gives $Mx = (1/\rho)Nx$. Therefore, we obtain

$$0 \leqslant Ax = M(I - M^{-1}N)x = (1-\rho)Mx = \frac{1-\rho}{\rho}Nx$$

with a positive factor $(1-\rho)/\rho$ since $\rho < 1$. So $Nx \geqslant 0$, and equality is excluded because otherwise $Ax = 0$ with $x \neq 0$ which is impossible. □

We now turn to the announced generalization of Theorem 2.1.

Theorem 3.5. *Assume that $A_1 = M_1 - N$, $A_2 = M_2 - N$ are two weak nonnegative splittings of different types of nonsingular square matrices A_1, A_2 with $N \neq O$. Assume that $A_1 \lneqq A_2$ (or, equivalently, $M_1 \lneqq M_2$) and that $A_1^{-1} > O$, $A_2^{-1} \geqslant O$. Then $\rho(M_2^{-1}N) < \rho(M_1^{-1}N) < 1$.*

Proof. By Theorem 3.4(i) we know that $\rho(M_1^{-1}N) < 1$ and $\rho(M_2^{-1}N) < 1$. Denote $G_1 = A_1^{-1}N$, $G_2 = A_2^{-1}N$ and $\tilde{G}_1 = NA_1^{-1}$, $\tilde{G}_2 = NA_2^{-1}$. We have

$$G_i = A_i^{-1}N = (I - M_i^{-1}N)^{-1}(M_i^{-1}N), \quad i = 1, 2,$$
$$\tilde{G}_i = NA_i^{-1} = NM_i^{-1}(I - NM_i^{-1})^{-1}, \quad i = 1, 2.$$

Now first assume that $A_1 = M_1 - N$ is of first type whereas $A_2 = M_2 - N$ is of second type. We thus have $M_1^{-1}N \geqslant O$ and $NM_2^{-1} \geqslant O$ so that G_1 and $\tilde{G}_2$ are nonnegative matrices. Since $\rho(G_i) = \rho(\tilde{G}_i)$ and since the function $f : t \mapsto t/(1-t)$ is strictly increasing on $[0, 1)$, we have

$$\begin{aligned}\rho(G_i) = \rho(\tilde{G}_i) &= \rho\big(M_i^{-1}N\big)/\big(1 - \rho\big(M_i^{-1}N\big)\big)\\ &= \rho\big(NM_i^{-1}\big)/\big(1 - \rho\big(NM_i^{-1}\big)\big)\end{aligned}$$

for $i = 1, 2$. Thus all we need to show is $\rho(G_2) < \rho(G_1)$. By Theorem 3.4(ii) there exists a vector $x \gneqq 0$ such that $M_2^{-1}Nx = \rho(M_2^{-1}N)x$ and $Nx \gneqq 0$. By Lemma 3.1 we have $A_1^{-1} > A_2^{-1}$, so that together with $A_1^{-1} > O$ we get

$$G_1x = A_1^{-1}Nx > A_2^{-1}Nx = G_2x = \rho(G_2)x. \tag{8}$$

Whence $\rho(G_2) < \rho(G_1)$ by Lemma 3.2(i).

If $NM_1^{-1} \geqslant O$ and $M_2^{-1}N \geqslant O$ then $\tilde{G}_1$ and G_2 are nonnegative matrices. Again by Theorem 3.4(ii) there exists a nonzero nonnegative vector z such that $M_1^{-1}Nz = \rho(M_1^{-1}N)z$ and $Nz \gneqq 0$. Thus

$$\rho(G_1)z = G_1z = A_1^{-1}Nz > A_2^{-1}Nz = G_2z, \tag{9}$$

and we obtain $\rho(G_2) < \rho(G_1)$ by Lemma 3.2(ii). □

As a first comment, let us note that in the proof we only made use of the inequality $A_1^{-1} - A_2^{-1} > O$, but not of $A_1 \lneqq A_2$. In the light of Lemma 3.1 a slightly more general version of the theorem therefore arises if one replaces the assumption $A_1 \lneqq A_2$ by $A_1^{-1} - A_2^{-1} > O$.

In the following corollaries we now emphasize two special cases of Theorem 3.5. In particular, these corollaries resemble Theorem 2.1 where the hypothesis of M-splittings has been replaced with the hypothesis of either regular or nonnegative splittings.

Corollary 3.6. *Assume that* $A_1 = M_1 - N$, $A_2 = M_2 - N$ *are two regular splittings of square nonsingular matrices* A_1, A_2 *with* $N \neq O$. *Moreover, let* $A_1 \lneqq A_2$ (*or, equivalently,* $M_1 \lneqq M_2$) *and assume that* $A_1^{-1} > O$, $A_2^{-1} \geqslant O$. *Then* $\rho(M_2^{-1}N) < \rho(M_1^{-1}N) < 1$.

Corollary 3.7. *Assume that* $A_1 = M_1 - N$, $A_2 = M_2 - N$ *are two nonnegative splittings of square nonsingular matrices* A_1, A_2 *with* $N \neq O$. *Moreover, let* $A_1 \lneqq A_2$ (*or, equivalently,* $M_1 \lneqq M_2$) *and assume that* $A_1^{-1} > O$, $A_2^{-1} \geqslant O$. *Then* $\rho(M_2^{-1}N) < \rho(M_1^{-1}N) < 1$.

We comment now on how some of the hypotheses of the results in this section cannot be weakened.

For regular splittings, we cannot weaken the assumption $A_1^{-1} > O$ by replacing it by the hypothesis that A_1^{-1} is irreducible, for in Example 3.8 below A_1^{-1} is irreducible and indeed has precisely one element equal to 0.

Example 3.8. Let

$$M_1 = \begin{pmatrix} 20 & -10 & -10 \\ -10 & 15 & -9 \\ -10 & 5 & 15 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 20 & -10 & -10 \\ -10 & 15 & -6 \\ -10 & 5 & 15 \end{pmatrix},$$
$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and put $A_1 = M_1 - N$, $A_2 = M_2 - N$. Here M_2 has just one positive off-diagonal element. Then, as we show explicitly below, M_2^{-1}, M_1^{-1}, and A_1^{-1} are nonnegative and only the (3, 2)-element is 0 in each case.

$$A_1^{-1} = \frac{1}{80}\begin{pmatrix} 12 & 4 & 12 \\ 11 & 8 & 14 \\ 5 & 0 & 10 \end{pmatrix},$$
$$M_1^{-1} = \frac{1}{200}\begin{pmatrix} 27 & 10 & 24 \\ 24 & 20 & 28 \\ 10 & 0 & 20 \end{pmatrix}, \quad M_2^{-1} = \frac{1}{400}\begin{pmatrix} 51 & 20 & 42 \\ 42 & 40 & 44 \\ 20 & 0 & 40 \end{pmatrix}.$$

But $\rho(M_1^{-1}N) = \rho(M_2^{-1}N) = 1/5$.

4. Further results

The purpose of this section is to formulate additional comparison results for splittings $A_1 = M_1 - N_1$, $A_2 = M_2 - N_2$ with two possibly different matrices N_1, N_2. Before we do so, we take a closer look at known results for two splittings of a *single* matrix A which we summarize in the following theorem.

Theorem 4.1. *Assume that A is a nonsingular matrix such that $A^{-1} > O$ and let $A = M_1 - N_1$ and $A = M_2 - N_2$ be two splittings of A. Then $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$ in the following cases*:

(i) *both splittings are regular*, $N_1 \neq 0$ *and* $M_1 \lneq M_2$ *or*, *equivalently*, $N_1 \lneq N_2$ [15, Theorem 3.32];
(ii) *both splittings are regular and* $M_1^{-1} > M_2^{-1}$ [(16, Theorem 3.6], *see also* [4]);
(iii) *both splittings are weak nonnegative splittings of different types and* $M_1^{-1} > M_2^{-1}$ [3, Theorem 7].

Clearly, Part (iii) contains the other two as well as an additional comparison result by Elsner [5] where one splitting is regular and the other is weak nonnegative of first type.

Note that Part (i) uses the same hypothesis $M_1 \lneq M_2$ as we used in Theorem 3.5, but it establishes the *reverse* inequality between the spectral radii. This is no contradiction though, since in Theorem 3.5 the assumption $A_1 \lneq A_2$ excludes equality between A_1 and A_2.

Taking Theorem 4.1 as a source of inspiration, and as was done in [8,10], we will now formulate comparison theorems for two splittings $A_1 = M_1 - N_1$, $A_2 = M_2 - N_2$ of different matrices with similar hypotheses as in Parts (i) and (iii). As before, the emphasis is on strict inequality of the spectral radii, without always having strict inequalities in the hypotheses. As it turns out, requiring $M_2 - M_1 \gneq O$ or $M_1^{-1} - M_2^{-1} > O$ does not suffice. We need to bound these differences by $A_2 - A_1$ and $A_1^{-1} - A_2^{-1}$, respectively.

Theorem 4.2. *Assume that $A_1 = M_1 - N_1$, $A_2 = M_2 - N_2$ are two weak nonnegative splittings of different types of nonsingular matrices A_1, A_2 with $N_1, N_2 \neq O$. Assume that $A_1^{-1} - A_2^{-1} > O$ and that $A_1^{-1} > O$, $A_2^{-1} \geqslant O$.*

(i) *If*

$$M_2 - M_1 \leqslant A_2 - A_1, \tag{10}$$

then

$$\rho\big(M_2^{-1}N_2\big) < \rho\big(M_1^{-1}N_1\big) < 1. \tag{11}$$

(ii) *If* $\rho(M_1^{-1}N_1) > 0$ *and*

$$M_1^{-1} - M_2^{-1} \geqslant A_1^{-1} - A_2^{-1}, \tag{12}$$

then

$$\rho\big(M_1^{-1}N_1\big) < \rho\big(M_2^{-1}N_2\big) < 1. \tag{13}$$

Proof. By Theorem 3.4(i) we know that $\rho_1 = \rho(M_1^{-1}N_1) < 1$, $\rho_2 = \rho(M_2^{-1}N_2) < 1$. So we only have to prove the first inequalities in (11) and (13). To prove Part (i) note first that (10) is equivalent to $N_1 \geqslant N_2$. We can therefore repeat the proof for Theorem 3.5 step by step, except that in (8) we have to use one additional inequality to obtain

$$G_1x = A_1^{-1}N_1x > A_2^{-1}N_1x \geqslant A_2^{-1}N_2x = G_2x = \rho(G_2)x,$$

and similarly for (9).

To prove Part (ii), assume first that $A_1 = M_1 - N_1$ is of second type and $A_2 = M_2 - N_2$ is of first type, so that $N_1M_1^{-1} \geqslant O$ and $M_2^{-1}N_2 \geqslant O$. Using (12) we obtain

$$\begin{aligned} M_2^{-1}N_2A_2^{-1} &= M_2^{-1}(M_2 - A_2)A_2^{-1} = A_2^{-1} - M_2^{-1} \\ &\geqslant A_1^{-1} - M_1^{-1} = M_1^{-1}(M_1 - A_1)A_1^{-1} \\ &= M_1^{-1}N_1A_1^{-1} \\ &= A_1^{-1}N_1M_1^{-1} \geqslant O. \end{aligned}$$

Now let $x \gneq 0$ and $y \gneq 0$ be two vectors such that

$$N_1M_1^{-1}x = \rho_1x, \quad y^{\mathrm{T}}M_2^{-1}N_2 = y^{\mathrm{T}}\rho_2.$$

These exist by standard Perron–Frobenius theory. Thus

$$\rho_2y^{\mathrm{T}}A_2^{-1}x = y^{\mathrm{T}}M_2^{-1}N_2A_2^{-1}x \geqslant y^{\mathrm{T}}A_1^{-1}N_1M_1^{-1}x = \rho_1y^{\mathrm{T}}A_1^{-1}x.$$

Since by assumption $A_1^{-1} > A_2^{-1}$, and since x and y are both nonzero and $\rho_1 > 0$ we obtain

$$\rho_2y^{\mathrm{T}}A_2^{-1}x > \rho_1y^{\mathrm{T}}A_2^{-1}x.$$

Therefore

$$\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2).$$

The case $M_1^{-1}N_1 \geqslant O$, $N_2M_2^{-1} \geqslant O$ can be proved in a similar way. □

As a first comment, let us note that a special case of the assumption $A_1^{-1} - A_2^{-1} > O$ arises when $A_1 \lneq A_2$ as we know from Lemma 3.1.

Next, we note that Part (i) of the above theorem generalizes Theorem 3.5 since in the case $N_1 = N_2$ the assumption (10) is automatically fulfilled (and equality holds there).

Let us further stress the fact that Part (ii) establishes $\rho(M_1^{-1}N_1)$ as the *smaller* quantity, in contrast to our previous results in Section 3 and to Part (i). So we consider this part as being much more related to the classical single splitting case of Theorem 4.1(iii). Since the conclusions in both parts of Theorem 4.2 are incompatible, the theorem also shows that the respective hypotheses are mutually exclusive. This can essentially also be seen directly: For weak nonnegative splittings of either type we have $\rho(M_i^{-1}N_i) = \rho(N_iM_i^{-1}) < 1$ so that

$$A_i^{-1} = \sum_{\nu=0}^{\infty} \left(M_i^{-1} N_i\right)^{\nu} M_i^{-1} = M_i^{-1} \sum_{\nu=0}^{\infty} \left(N_i M_i^{-1}\right)^{\nu},$$

which shows that $O \leqslant M_i^{-1} \leqslant A_i^{-1}$ for $i = 1, 2$. Therefore, if we assume (10), i.e., $M_2 - M_1 \leqslant A_2 - A_1$ we obtain

$$\begin{aligned} M_1^{-1} - M_2^{-1} &= M_2^{-1}(M_2 - M_1)M_1^{-1} \\ &\leqslant A_2^{-1}(M_2 - M_1)A_1^{-1} \\ &= A_2^{-1}(A_2 - A_1)A_1^{-1} \\ &= A_1^{-1} - A_2^{-1}, \end{aligned}$$

which is essentially the opposite of (12).

We now discuss examples which show that Theorem 4.2 no longer holds when certain hypotheses are relaxed. First of all we note that the hypothesis $A_1^{-1} > O$ cannot be relaxed in Part (i), Example 3.8 representing a counter-example.

Also, the following example shows that without bounds of the kind (10) and (12) we cannot expect comparison results for the spectral radii.

Example 4.3. Let

$$\begin{aligned} A_1 &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = M_1 - N_1, \\ A_2 &= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} = M_2 - N_2, \\ A_3 &= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = M_3 - N_3. \end{aligned}$$

All these splittings are M-splittings and

$$A_1^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > A_2^{-1} = A_3^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and

$$M_1^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} > M_3^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} > M_2^{-1} = \frac{1}{15}\begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix}.$$

We have $M_2 - M_1 \geqslant O$ but $M_2 - M_1 \nleqslant A_2 - A_1$ as well as $M_3 - M_1 \geqslant O$ but $M_3 - M_1 \nleqslant A_3 - A_1$. So the pairs given by the first and second splittings and by the first and third splittings satisfy all hypotheses of Theorem 4.2, except (10). But we have $\rho(M_2^{-1}N_2) = 4/5 < \rho(M_1^{-1}N_1) = 2/3 > \rho(M_3^{-1}N_3) = 4/7$, showing that either inequality between the spectral radii may now occur. Note that we also have $M_1^{-1} - M_2^{-1} > O$ but $M_1^{-1} - M_2^{-1} \ngeqslant A_1^{-1} - A_2^{-1}$ as well as $M_1^{-1} - M_3^{-1} > O$ but

$M_1^{-1} - M_3^{-1} \not\geqslant A_1^{-1} - A_3^{-1}$, i.e., neither pair of splittings satisfies (12). Actually, we have $M_1^{-1} - M_2^{-1} \leqslant A_1^{-1} - A_2^{-1}$ and $M_1^{-1} - M_3^{-1} \leqslant A_1^{-1} - A_3^{-1}$. This shows that a modification of Part (ii) with reversed inequalities in (12) and (13) does not hold.

Example 4.3 can be modified slightly to show that Part (ii) of Theorem 4.2 is not empty, i.e., that all hypotheses there can be met. To this purpose, subtract a small positive quantity ε from the $(1, 2)$ entry of A_3 and add it at the same position in N_3. If ε is small enough, we will get $M_3^{-1} - M_2^{-1} \geqslant A_3^{-1} - A_2^{-1}$ and $A_3 \lneqq A_2$ with $A_3^{-1} > O$, $A_2^{-1} \geqslant O$ and both splittings are M-splittings.

Remark 4.4. (i) Interestingly, there exists an obvious, but also new, counterpart of Theorem 4.2 with all strict inequalities replaced by nonstrict ones. (ii) It is possible to formulate a version of Theorem 4.2 where some inequalities between matrices are replaced by less restrictive inequalities between certain matrix vector products. The basic idea can be caught from Theorem 4.5 below, so we do not give details here.

We finish with an additional result where the nonnegativity assumptions on one of the splittings are kept to what we think is a minimum.

Theorem 4.5. *Assume that $A_1 = M_1 - N_1$, $A_2 = M_2 - N_2$ are two splittings of nonsingular matrices A_1, A_2 with $N_1, N_2 \neq O$. Let $M_2^{-1}N_2 \geqslant O$ and assume that there exists an eigenvector $w \geqslant 0$ corresponding to $\rho(M_1^{-1}N_1)$ such that $0 \leqslant A_1 w$ and*

$$A_1 w < A_2 w. \tag{14}$$

Moreover, let $O \leqslant M_1^{-1} \leqslant M_2^{-1}$. Then

$$\rho\big(M_2^{-1}N_2\big) < \rho\big(M_1^{-1}N_1\big). \tag{15}$$

Proof. The hypotheses allow us to establish the following chain of (in)equalities:

$$\begin{aligned}\rho\big(M_1^{-1}N_1\big)w &= M_1^{-1}N_1 w = w - M_1^{-1}A_1 w \geqslant w - M_2^{-1}A_1 w \\ &> w - M_2^{-1}A_1A_1^{-1}A_2 w = (I - M_2^{-1}A_2)w = M_2^{-1}N_2 w.\end{aligned}$$

Since, by assumption, $M_2^{-1}N_2 \geqslant O$ this gives $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$ by Lemma 3.2(ii). □

Theorem 4.5 extends Lemma 2.2 of [10]; cf. also Theorems 3.13 and 3.15 of [8]. The following example shows that this theorem is not contained in our previous ones.

Example 4.6. Let

$$A_1 = \begin{pmatrix} -1 & 4/3 \\ 4/3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 4/3 \\ 4/3 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M_1 - N_1$$

and

$$A_2 = \begin{pmatrix} 1 & -5/8 \\ -5/8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -5/9 \\ -5/9 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 5/72 \\ 5/72 & 0 \end{pmatrix} = M_2 - N_2.$$

Then

$$0 \leqslant M_1^{-1} = \begin{pmatrix} 0 & 3/4 \\ 3/4 & 0 \end{pmatrix} \leq \begin{pmatrix} 81/56 & 45/56 \\ 45/56 & 81/56 \end{pmatrix} = M_2^{-1}.$$

So both splittings are regular splittings. Moreover, $w = (1, 1)^T$ is a positive eigenvector corresponding to the eigenvalue 3/4 (which is the spectral radius) of

$$M_1^{-1}N_1 = \begin{pmatrix} 0 & 3/4 \\ 3/4 & 0 \end{pmatrix}$$

Finally, $0 \leqslant A_1 w < A_2 w$. So all assumptions of Theorem 4.5 are met, while the condition $M_2 - M_1 \leqslant A_2 - A_1$ of Theorem 4.2(i) is violated. We have $\rho(M_2^{-1}N_2) = 5/32$, $\rho(M_1^{-1}N_1) = 3/4$.

We have some observations concerning the hypotheses of Theorem 4.5. We assume that the eigenvector $w \geqslant 0$, but the proof shows that the hypotheses imply $w > 0$. Moreover, since $0 \leqslant A_1 w = (M_1 - N_1)w = M_1(I - M_1^{-1}N_1)w = (1 - \rho(M_1^{-1}N_1))M_1 w$ and $M_1^{-1} \geqslant 0$, we get $\rho(M_1^{-1}N_1) < 1$; see [12, Lemma 1] for a similar argument in the special case of a weak nonnegative splitting of first type. By (15), we also have $\rho(M_2^{-1}N_2) < 1$ which is equivalent to $A_2^{-1} \geqslant O$ (see [15, Theorem 3.37]), because the splitting $A_2 = M_2 - N_2$ is weak nonnegative of first type.

Note that (14) is fulfilled if $A_2 - A_1 \geqslant O$ contains no zero rows. Note also that by Theorem 3.4(ii) a vector $w \gneqq 0$ such that $A_1 w \gneqq 0$ exists if the splitting $A_1 = M_1 - N_1$ is weak nonnegative of second type.

In Theorem 4.5 the smaller spectral radius corresponds to the larger matrix M_2^{-1}. But M_2^{-1} belongs to the splitting of the "smaller" matrix A_2^{-1}, as the remarks above show. So, as illustrated by Example 4.6, this is yet another situation, different from that of Theorem 4.2. Together with Theorem 3.5 our theorems show the variety of possible results if one compares splittings of *different* matrices.

Remark 4.7. Comparison results for splittings of different matrices have previously been used in several ways to study (nonstationary) iterative methods for a (single) system of equations. Typically, one then models the iteration by "macro-iterations" involving splittings of different "macro-matrices". Examples include the study of multisplittings in [10] (and several subsequent publications) and the paper [6], where the effect of the granularity in the block Jacobi method on its asymptotic rate of convergence is studied. Comparison results on splittings of different matrices appear as an important tool for the analysis in these situations; see, e.g., [[10], Lemma 2,2]. The comparison results of the present paper are an attempt to develop such comparison results in a systematic manner. We believe that they will be useful as a tool in further investigations on nonstationary iterative processes, including, e.g., inner-outer iterations.

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