Inexact Newton with Krylov projection and recycling for Riccati equations

Marlindy Monsalve and Daniel B. Szyld
Department of Mathematics, Temple University, Philadelphia
2010 DOE Applied Mathematics Program Meeting, 3-5 May 2010

Abstract
Our purpose is to solve Riccati equations which arise in many applications in Control Theory of the form
\[ AX + XA^T - XBX + C^T C = 0, \]
where \( A, X, B, C \in \mathbb{R}^{n \times n} \). It is standard to seek low-rank solutions of the form \( X = X_1Z \), \( Z \in \mathbb{R}^{n \times m} \) with \( m \ll n \) (for storage issues). Since this is a nonlinear matrix equation, Newton’s method is a successful strategy: it requires the solution of a Lyapunov equation at each iteration \( j \), given by
\[ M_sS + S^T M_s^T - D_sD_s^T = 0, \]
where \( M_s = A - X \cdot B^T \) and \( D_s^T = C^T \cdot X \cdot B \).

In this work in progress, we propose to use an Inexact Newton method to solve (1) and a Krylov projection method to solve (2) with a recycling process.

Introduction
Several general schemes to solve \( F(X) = 0 \) when \( F \) is a Riccati equation

Krylov Projection Method
\[ MS + SM^T - D^2 = 0. \]

Find \( K \) such that \( \text{Krylov subspace} \ K^0(M,t) \)
\[ T = V V^T = I. \]

The effectiveness of these methods depend on the subspace \( K^0(M,t) \). There are two attractive approaches to find \( K^0(M,t) \):

1. Inexact Newton + Krylov Projection
2. Recycling ideas: take advantage of the Krylov subspace obtained in the j-th step to enrich the Krylov subspace of step \( j+1 \)

Global Arnoldi
1. Use Frobenius norm and its inner product.
2. Produce \( \{v_m \} \) such that
\[ M_s v_m + v_m^T M_s^T = \phi_{\text{Frobenius}}(X, M_s) \]
3. The small Lyapunov equation \( H_j, X_{i+1}^T, x_{i+1} \)
4. The dimension of \( H_j \) is \( m \)

Extended Krylov
1. Use Euclidean norm and its inner product.
2. Produce \( \{v_m \} \) such that
\[ M_s v_m + v_m^T M_s^T = \phi_{\text{Euclidean}}(X, M_s) \]
3. The small Lyapunov equation \( H_j, X_{i+1}^T, x_{i+1} \)
4. The dimension of \( H_j \) is \( m \)

Recycling ideas: Take advantage of the Krylov subspace obtained in the j-th step to enrich the Krylov subspace of step \( j+1 \). (idea based on [5]). This is done with the aim of

1. Accelerating the convergence in the j-th step.
2. Saving storage and computational effort.

The work already done to create the basis at the j-th step is not thrown away; it enriches the space at the next step. In Newton for Riccati, the matrix \( M \) is not too far from \( M_j \) (conceptually).

Inexact Newton Method
for \( j = 1, 2, \ldots \)
(i) \( M_j = A - X \cdot B^T \)
(ii) \( D_j = C^T \cdot X \cdot B \)
(iii) Find \( S_j \) such that \( M_j S_j + S_j M_j^T - D_j D_j^T = 0 \)
end

Krylov projection method for Lyapunov equations: Roughly speaking, this approach to solve (2) consists in

1. Project the Lyapunov equation onto a Krylov subspace \( K^0(M_j) \) of dimension \( m \)
2. The approximate solution of \( (2) \) will be \( X = V V^T \), where \( V \) is an orthogonal basis of \( K^0(M_j) \) and \( V \) is the solution of a small Lyapunov equation (dimension \( m \)).

References