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Existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods

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Summary. Given a nonsingular matrix A, and a matrix T of the same order, under certain very mild conditions, there is a unique splitting $A = B - C$, such that $T = B^{-1}C$. Moreover, all properties of the splitting are derived directly from the iteration matrix T. These results do not hold when the matrix A is singular. In this case, given a matrix T and a splitting $A = B - C$ such that $T = B^{-1}C$, there are infinitely many other splittings corresponding to the same matrices A and T, and different splittings can have different properties. For instance, when T is nonnegative, some of these splittings can be regular splittings, while others can be only weak splittings. Analogous results hold in the symmetric positive semidefinite case. Given a singular matrix A, not for all iteration matrices T there is a splitting corresponding to them. Necessary and sufficient conditions for the existence of such splittings are examined. As an illustration of the theory developed, the convergence of certain alternating iterations is analyzed. Different cases where the matrix is monotone, singular, and positive (semi)definite are studied.

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1. Introduction and preliminaries

Consider the solution of linear systems of the form

$$Ax = b, \tag{1}$$

where A is a square matrix of order n, possibly singular, and $x, b \in \mathbb{R}^n$. The representation $A = B - C$ is called a splitting if B is nonsingular. A splitting gives rise to the classical iterative method

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$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \ldots, \tag{2}$$

where $T = B^{-1}C$ is called the iteration matrix of the method, $c = B^{-1}b$, and $x^0 \in \mathbb{R}^n$ is given as the initial guess. We denote by $T \geq 0$ a nonnegative matrix, i.e., a matrix with nonnegative entries, by $\sigma(T)$ the spectrum of T, and by $\rho(T)$ its spectral radius. By I we denote the identity matrix of order n, and by I_r that of order r for $r \neq n$.

It is well-known that the convergence of the method (2), i.e., of the sequence $\{x^k\}$, depends on the convergence of the sequence T^k as $k \to \infty$; see, e.g., [28]. Following [17] and other authors, we say that T is *convergent* if the powers T^k converge to a limiting matrix as $k \to \infty$. If that limit is the zero matrix, T is called *zero-convergent*. As is well-known, for A nonsingular a necessary and sufficient condition for the convergence of (2) for any x^0 is that T be zero-convergent, or, equivalently, that $\rho(T) < 1$. In the singular case the situation is more involved; see, e.g., [4], [26]. In this case, $1 \in \sigma(T)$ and a necessary condition for convergence is that $\rho(T) = 1$ be the only eigenvalue in the unit circle, i.e., that $\gamma(T) := \max\{|\lambda|,\ \lambda \in \sigma(T),\ \lambda \neq 1\} < 1$. If (1) is consistent and T is convergent, the iterative scheme (2) converges to a solution of (1) which depends, in general, on the initial guess x^0.

A real, not necessarily symmetric matrix C is *positive definite* if $x^{\mathrm{T}}Cx > 0$ for all real $x \neq 0$. This is equivalent to requiring that the symmetric part of C, denoted $C^{\mathrm{S}} := (C + C^{\mathrm{T}})/2$, be positive definite in the usual sense.

Definition 1.1. Let $A = B - C$ be a splitting, and $T = B^{-1}C$ the corresponding iteration matrix. The splitting is called *P-regular* if $B + C$ is positive definite [21], *weak* if $T \geq 0$ [15], *weak regular* if $B^{-1} \geq 0$ and $T \geq 0$ [4], *regular* if $B^{-1} \geq 0$ and $C \geq 0$ [28], and an *M-splitting* if B is an M-matrix and $C \geq 0$ [24]. A nonsingular M-matrix can be defined as having nonpositive off-diagonal elements and its inverse being nonnegative; see, e.g., [28].

Note that for A symmetric, the splitting $A = B - C$ being P-regular is equivalent to requiring that $B + B^{\mathrm{T}} - A$ be positive definite in the usual sense.

The classifications in Definition 1.1 have been used as important tools to obtain convergence results of the methods of the form (2). In particular, we have the following two convergence results for monotone and symmetric positive definite matrices, respectively. A nonsingular matrix A is called *monotone* if $A^{-1} \geq 0$.

Lemma 1.2. [4] *Let $A = B - C$ be a weak regular splitting, and let $T = B^{-1}C$. Then $\rho(T) < 1$ if and only if the matrix A is monotone.*

Lemma 1.3. [21] *Let $A = B - C$ be a P-regular splitting of the symmetric matrix A, and let $T = B^{-1}C$. Then $\rho(T) < 1$ if and only if A is positive definite.*

For the case of nonsingular A, the following simple result was used, e.g., in [13], to obtain a splitting *induced* by a given iteration matrix, and to then consider the characteristics of the induced splitting to study convergence.

Lemma 1.4. [13] *Let A and T be square matrices such that A and $I - T$ are nonsingular. Then, there exists a unique pair of matrices B, C, such that B is nonsingular, $T = B^{-1}C$ and $A = B - C$. The matrices are $B = A(I - T)^{-1}$ and $C = B - A$.*

This Lemma is very useful when analyzing parallel algorithms, since the application of many operators simultaneously can be rewritten as a single splitting whose properties can then be analyzed; see, e.g., [20] or [29], where the Lemma was implicitly used. It is also useful for studying the convergence of two-stage methods [12] and of alternating iterations; see Sect. 3. We point out that the constructive properties of Lemma 1.4, though simple and very useful, were not always appreciated; see, e.g., [18, Sect. 2] where it is mentioned that the conclusions of the Lemma do not hold.

The following observation, which is a little surprising at first, is a direct consequence of the uniqueness in Lemma 1.4.

Remark 1.5. Let A and T satisfy the hypothesis of Lemma 1.4. The characteristics of the unique splitting induced by T are intrinsically already given, i.e., if the splitting is weak, weak regular, etc. this property is already imbedded in the structure of A and T.

In this paper we further consider iterative methods of the form (2) when the matrix A is singular and the system (1) is solvable, e.g., when one is looking for the stationary probability distribution of a Markov chain [4], or when solving discretized elliptic partial differential equations with Neumann or periodic boundary conditions [23]. As it is shown in Sect. 2, in the singular case, the results analogous to Lemma 1.4 and Remark 1.5 are totally opposite to those in the nonsingular case: there are infinitely many splittings induced by the same iteration matrix, and different splittings induced by the same matrices can have different properties.

The results in this paper deal with the existence and uniqueness of the splittings. The existence question, sometimes referred to as the consistency question, has been studied before, e.g., in [3], [19], [27], [31]. In Sect. 2 we present a constructive proof.

In Sects. 3 and 4, we analyze in detail certain alternating iterative methods. The theory developed in the first part plays a crucial role in this analysis. We show, for example, that if the different splittings defining each iteration are P-regular, there is overall convergence, and moreover, the induced splitting is also P-regular. This holds both in the symmetric positive definite case, as well as in the semidefinite case. Finally, a comparison result is presented which shows that under certain conditions, the asymptotic convergence rate of the alternating method is at least as fast as the faster single iterative method defining each iteration.

A longer version of this paper, with extensive references and more examples can be found in [2].

2. Splittings of singular matrices

By $\mathcal{N}(M)$ and $\mathcal{R}(M)$ we denote the null space and the range of the matrix M, respectively. We begin this section with the existence question, cf. Lemma 1.4.

Theorem 2.1. *Let A and T be $n \times n$ matrices. The necessary and sufficient condition for the existence of a splitting $A = B - C$ with $T = B^{-1}C$ is that*

$$\mathcal{N}(A) = \mathcal{N}(I - T). \tag{3}$$

Proof. The necessity follows from the fact that if $A = B - C$ with B nonsingular, we have $A = B(I - T)$. For the sufficiency, let $r = \text{rank}(A) = \text{rank}(I - T)$. Let $a_{i_1}, a_{i_2} \ldots, a_{i_r}$ be r linearly independent columns of A. From the hypothesis (3) it follows that $h_{i_1}, h_{i_2} \ldots, h_{i_r}$ are linearly independent columns of $H = I - T$. Let $v_j = a_{i_j}$, $w_j = h_{i_j}$, $j = 1, 2, \ldots, r$. Let $v_{r+1}, \ldots, v_n$ a basis of $\mathcal{N}(A^{\mathrm{T}}) = \mathcal{R}(A)^{\perp}$ and $w_{r+1}, \ldots, w_n$ a basis of $\mathcal{R}(I - T)^{\perp}$. The nonsingular matrix B we are looking for can be defined by

$$Bw_j = v_j \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{4}$$

In other words, let $V = [v_1, v_2, \ldots, v_n]$ and $W = [w_1, w_2, \ldots, w_n]$, then chose $B = VW^{-1}$. To see that the matrix B thus constructed satisfies

$$A = B(I - T), \tag{5}$$

we look at this equality one column at a time. If $k = i_j$ for some j, i.e., if a_k belongs to the chosen basis of $\mathcal{R}(A)$, the equality (5) follows from the definition of B in (4). If a_k is not in the chosen basis of $\mathcal{R}(A)$, we write $a_k = \sum_{j=1}^{r} \alpha_j v_j = B \sum_{j=1}^{r} \alpha_j w_j = Bh_k$, where the last equality follows from the hypothesis (3). □

Theorem 2.1 is not new. It can be found, e.g., in [3], [30], [31]. The proof here, unlike that of Young, does not use the Jordan form of the matrix. Note also that different choices of the vectors $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ in the proof can produce different splittings of a singular matrix A. This leads to the following non-uniqueness theorem.

Theorem 2.2. *Let $A \in \mathbb{R}^{n \times n}$ be a singular matrix of rank $n - s < n$. If $A = B - C$ is a splitting, then there are infinitely many other splittings $A = F - G$ with the same iteration matrix, i.e., with $T = B^{-1}C = F^{-1}G$. The matrices F have the form $F = (B^{-1} + UV^{\mathrm{T}})^{-1}$, where $V \in \mathbb{R}^{n \times r}$ is a matrix whose columns belong to $\mathcal{N}(A^{\mathrm{T}})$, $r \leq s$, and $I_r + V^{\mathrm{T}}BU$ is nonsingular. Conversely, if two splittings $A = B - C = F - G$ are such that they have the same iteration matrix, then there exists a matrix $U \in \mathbb{R}^{n \times s}$ such that $F = (B^{-1} + UV^{\mathrm{T}})^{-1}$, where $V \in \mathbb{R}^{n \times s}$ is a matrix whose columns belong to $\mathcal{N}(A^{\mathrm{T}})$ and $I_s + V^{\mathrm{T}}BU$ is nonsingular. The rank of the difference $B - F$ is at most s.*

Proof. Let $s = \dim \mathcal{N}(A^{\mathrm{T}}) = \dim \mathcal{N}(A)$. Let $0 \neq V \in \mathbb{R}^{n\times r}$ be a matrix whose columns lie in $\mathcal{N}(A^{\mathrm{T}})$, $r \leq s$, and let U be any $n \times r$ matrix such that $I_r + V^{\mathrm{T}}BU$ is nonsingular (note that it is always possible to find a nonzero matrix U with this property). Let $F = M^{-1}$, and $G = F - A$, where $M = B^{-1} + UV^{\mathrm{T}}$. The nonsingularity of M can be established using the Sherman-Morrison-Woodbury formula, found, e.g., in [9]. It follows that $F^{-1}G = I - F^{-1}A = I - B^{-1}A = T$ and the first part of the Theorem is proved.

For the converse, from $F^{-1}A = B^{-1}A$ it follows that $A^{\mathrm{T}}(F^{-\mathrm{T}} - B^{-\mathrm{T}}) = 0$, that is, the columns of $F^{-\mathrm{T}} - B^{-\mathrm{T}}$ are in the null space of A^{T}:

$$(6) \qquad (F^{-\mathrm{T}} - B^{-\mathrm{T}})e_j \in \mathcal{N}(A^{\mathrm{T}}) \quad \text{for all} \quad j = 1, 2, \ldots, n,$$

where e_j is the jth column of the identity. Let $v_1, v_2, \ldots, v_s$ be a basis of $\mathcal{N}(A^{\mathrm{T}})$, then for each $1 \leq j \leq n$ there exist scalars $u_{j1}, u_{j2}, \ldots, u_{js}$ such that $(F^{-\mathrm{T}} - B^{-\mathrm{T}})e_j = \sum_{i=1}^{s} u_{ji} v_i$. Let now V be the $n \times s$ matrix whose ith column is v_i and U be the $n \times s$ matrix whose entry in position (i, j) is u_{ij}, then we can rewrite the last identity as $F^{-\mathrm{T}} - B^{-\mathrm{T}} = VU^{\mathrm{T}}$ or, equivalently, as $F^{-1} = B^{-1} + UV^{\mathrm{T}}$. From the Sherman-Morrison-Woodbury formula we get

$$(7) \qquad F = (B^{-1} + UV^{\mathrm{T}})^{-1} = B - BU(I_s + V^{\mathrm{T}}BU)^{-1}V^{\mathrm{T}}B.$$

The invertibility of $I_s + V^{\mathrm{T}}BU$ readily follows from the assumption that B and F are both invertible. Indeed, $F^{-1} = B^{-1} + UV^{\mathrm{T}} = B^{-1}(I + BUV^{\mathrm{T}})$ implies that $I + BUV^{\mathrm{T}}$ is nonsingular, that is, -1 is not an eigenvalue of BUV^{T}. Because the nonzero eigenvalues of BUV^{T} are the same as those of $V^{\mathrm{T}}BU$, we can conclude that $I_s + V^{\mathrm{T}}BU$ is also nonsingular.

Finally, from (6) it follows that the rank of $F^{-\mathrm{T}} - B^{-\mathrm{T}}$ is $r \leq s$, and thus, from (7) the difference between F and B is a matrix of rank r. In this case the matrix U can be chosen to have exactly $s - r$ zero columns, and the corresponding columns of V can be replaced by zero columns. Equivalently, such columns can just be deleted. □

Remark 2.3. In the context of Theorem 2.2, if $A = M - N$ is a weak splitting, so is any other splitting of A. But, as the following example shows, one can be a regular splitting (or even an M-splitting) while the other may not even be weak regular (only weak).

Example 2.4. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Consider

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}.$$

Let $N = M - A$, $G = F - A$, so that $T = M^{-1}N = F^{-1}G$. Note that $A = M - N$ is an M-splitting and thus a regular splitting, but $A = F - G$ is only weak. Also, notice that $A = M - N$ is a P-regular splitting of the symmetric positive semidefinite matrix A, whereas $A = F - G$ is not. Furthermore, $F^{-1} = M^{-1} - 2ee^{\mathrm{T}}$ where $e = (1,1)^{\mathrm{T}} \in \mathcal{N}(A^{\mathrm{T}})$, showing that the two matrices M and F differ by a rank-one matrix.

It is a consequence of the theory described so far that a splitting need not be regular or weak regular in order to obtain a convergent iteration (2). Thus, for example in Theorem 2 of [17], the hypothesis of regular splitting can be replaced with just being a weak splitting, and actually the proof in [17] carries through with no changes. This fact is also illustrated in Lemmas 4.4 and 4.5 of [16] where the hypothesis used is of having a weak splitting, and relates to the index of the eigenvalue 1 of the iteration matrix. This situation is also consistent with the observation in [16] and in [26] that condition (3) in the Theorem in [26] is independent of the splitting chosen. Another possible interpretation of the results presented is that for singular matrices, convergence follows only from the study of the iteration matrix, and not from the splitting. This is done, e.g., in [10, Sect. 2].

3. The convergence of alternating iterations

Consider the general class of iterative methods for the solution of (1) of the form

$$x^{k+1/2} = M^{-1}Nx^k + M^{-1}b, \quad x^{k+1} = P^{-1}Qx^{k+1/2} + P^{-1}b, \quad k = 0,1,\ldots, \tag{8}$$

where $A = M - N = P - Q$ are splittings of a possibly singular matrix A, and x^0 is the given initial guess. Many well-known methods belong to this class. When A is symmetric and $P = M^{\mathrm{T}}$, the first iteration of this kind is perhaps Aitken's *to-and-fro* method (symmetric Gauss-Seidel) [1]. Its extrapolated version is Sheldon's SSOR (Symmetric Successive Over-Relaxation) scheme [25]. One important feature of this method is that when A is symmetric positive definite (SPD), its convergence may be enhanced by Chebyshev or conjugate gradient (CG) acceleration. Another set of algorithms in the class described by (8) are the alternating direction implicit methods (ADI), see, e.g., [14], [30]. We also mention here the methods considered by Conrad and Wallach [5], [6], including SSOR and one alternating between a Gauss-Seidel and a Jacobi sweep. They presented efficient implementations on parallel computers with substantial operation savings; see also [22].

To analyze the convergence of the general scheme (8) we construct a single splitting $A = B - C$ associated with the iteration matrix. To that end, let us eliminate $x^{k+1/2}$ from (8) and obtain the iterative process

$$x^{k+1} = P^{-1}QM^{-1}Nx^k + P^{-1}(QM^{-1} + I)b, \quad k = 0,1,\ldots, \tag{9}$$

which is of the form (2), where $T = P^{-1}QM^{-1}N$. We use this formulation to study the convergence of (8) under the assumption that A is either monotone or symmetric and positive (semi)definite.

We note that the convergence of the individual splittings $A = M - N$ and $A = P - Q$ does not guarantee the convergence of the alternating iteration (8), as illustrated by the following example.

Example 3.1. Consider the symmetric positive definite M-matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and the splittings $A = M - N = P - Q$, where

$$M = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Both splittings are convergent, since $M^{-1}N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $P^{-1}Q = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Yet $T = P^{-1}QM^{-1}N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and therefore the iteration (8) is not convergent.

If A is nonsingular, we use Lemma 1.4 to obtain the unique splitting $A = B - C$ such that $B^{-1}C = T$. It is very natural to ask which properties of the splittings $A = M - N = P - Q$ are inherited by $A = B - C$. This question can be answered in a fairly complete manner together with sufficient conditions for the convergence of (8) if A is monotone or symmetric positive (semi)definite. We begin with the monotone case.

Theorem 3.2. *Let A be nonsingular, and $A^{-1} \geq 0$. If the splittings $A = M - N = P - Q$ are weak regular, then $T = P^{-1}QM^{-1}N$ is zero-convergent, and therefore the sequence $\{x^k\}$ generated by (8) converges to the unique solution of $Ax = b$ for any choice of the initial guess x^0. Furthermore, the unique splitting $A = B - C$ induced by T is weak regular.*

Proof. We will show that $\rho(T) < 1$. From

$$T = (I - P^{-1}A)(I - M^{-1}A) = I - P^{-1}A - M^{-1}A + P^{-1}AM^{-1}A$$

we find that

$$(I - T)A^{-1} = P^{-1} + M^{-1} - P^{-1}AM^{-1} = P^{-1} + (I - P^{-1}A)M^{-1}.$$

Because the splittings are weak regular, it follows that $T \geq 0$ and also $(I - T)A^{-1} \geq 0$. Hence $0 \leq (I + T + T^2 + \cdots + T^m)(I - T)A^{-1} = (I - T^{m+1})A^{-1} \leq A^{-1}$ for every nonnegative integer m. It follows, by a standard argument (see, e.g., [21]), that the partial sums of the series $\sum_{m=0}^{\infty} T^m$ remain uniformly bounded (in norm). Therefore, the series is convergent, and $\rho(T) < 1$.

Let $A = B - C$ be the unique splitting induced by $T = P^{-1}QM^{-1}N$, cf. Lemma 1.4 and (9). We have that

$$\text{(10)} \qquad \begin{aligned} B^{-1} &= P^{-1}(QM^{-1} + I) = P^{-1} + (I - P^{-1}A)M^{-1} \\ &= P^{-1}(M + P - A)M^{-1}. \end{aligned}$$

The nonsingularity of $M+P-A$ follows from that of $I-T$ and the identity

$$I-T=(P^{-1}+M^{-1}-P^{-1}AM^{-1})A=P^{-1}(M+P-A)M^{-1}A.$$

We see from (10) that $B^{-1}\geq 0$, and the proof is complete. □

Theorem 3.2 implies that weak regularity is a property which the single splitting $A=B-C$ inherits from the original splittings $A=M-N=P-Q$. The following example shows that if $A=M-N=P-Q$ are both regular splittings, the induced splitting $A=B-C$ need not inherit that property.

Example 3.3. Consider the M-matrix $A=\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$ and the splittings $A=M-N=P-Q$, where $M=\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $P=\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$. Both splittings are regular, but the unique splitting $A=B-C$ such that $B^{-1}C=T=P^{-1}QM^{-1}N$ is given by

$$B=\begin{bmatrix} 7/3 & -2/3 \\ -13/6 & 7/3 \end{bmatrix} \quad \text{and} \quad C=B-A=\begin{bmatrix} 1/3 & 1/3 \\ -1/6 & 1/3 \end{bmatrix}$$

and therefore it is not regular.

Let us consider now the case where A is singular. In this case, it follows from Theorems 2.1 and 2.2 that there are either infinitely many splittings corresponding to T, or none at all, depending on whether or not the compatibility condition (3) is satisfied. It turns out that, unlike in the nonsingular case, weak regularity of the splittings $A=M-N=P-Q$ does not imply compatibility. Consider, for example, the symmetric positive semidefinite M-matrix A from Example 2.4, with the weak regular splittings defined by $M=P=I$. Obviously, $M+P-A$ is singular, and the compatibility condition is violated. However, if we *impose* that $M+P-A$ be nonsingular, then the compatibility condition (3) holds, and the iteration matrix $T=P^{-1}QM^{-1}N$ is induced by the splitting

$$A=B-C\,,\quad \text{where} \quad B=P(M+P-A)^{-1}M\,. \tag{11}$$

It follows from (10) that $B^{-1}\geq 0$ and thus, this splitting is weak regular. Hence, we have proved the following result.

Theorem 3.4. *Let A be a singular matrix. If the splittings $A=M-N=P-Q$ are weak regular and $M+P-A$ is nonsingular, the splitting (11) is a weak regular splitting and $B^{-1}C=T=P^{-1}QM^{-1}N$.*

The splitting (11) is weak regular, but examples can be found to show that not all splittings corresponding to T are weak regular. We note that in many practical situations, the nonsingularity of $M+P-A$ is not difficult to check. Consider, as a simple example, the symmetric Gauss-Seidel method. If $A=L+D+U$ is the usual splitting of A into its lower triangular, diagonal, and upper triangular parts, then $M=L+D$, $P=D+U$ and $M+P-A=D$ is invertible if and only if A has no zeros on the main diagonal.

As is well-known, if the matrix A is singular, having a weak regular splitting is not sufficient for convergence of a method of the form (2), thus the hypotheses of Theorem 3.4 need to be supplemented so that $\gamma(T) < 1$. For example, if A is a singular, irreducible M-matrix, the results in [17] imply that $\rho(T) = 1$, and letting $T_\alpha := (1-\alpha)I + \alpha T$, we have $\gamma(T_\alpha) < 1$ for all $\alpha \in (0,1)$. It follows that the iteration (2) with T replaced by T_α is convergent to a solution of (1).

Next, we consider the symmetric positive (semi)definite case. There are many analogies, but also some interesting differences, with the monotone case. We already know, from Example 3.1, that convergence of the individual splittings $A = M - N$ and $A = P - Q$ of a symmetric positive definite matrix is not sufficient to insure convergence of the alternating iteration (8). However, if the splittings are P-regular, this property is inherited by the iteration matrix T of the combined iteration (2).

Theorem 3.5. *Let A be symmetric positive definite. If the splittings $A = M - N = P - Q$ are P-regular, then $T = P^{-1}QM^{-1}N$ is zero-convergent. Therefore, the sequence $\{x^k\}$ generated by (8) converges to the unique solution of $Ax = b$ for any choice of the initial guess x^0. Moreover, the unique splitting induced by the iteration matrix is P-regular.*

Proof. We show that $A - T^{\mathrm{T}}AT$ is positive definite (convergence will then follow from Stein's Theorem; see, e.g., [21]). Since the splittings $A = M - N = P - Q$ are P-regular, we know that $S := A - (P^{-1}Q)^{\mathrm{T}}A(P^{-1}Q)$ and $R := A - (M^{-1}N)^{\mathrm{T}}A(M^{-1}N)$ are both positive definite. Thus, the matrix $H := (M^{-1}N)^{\mathrm{T}}S(M^{-1}N)$ is positive semidefinite. But

$$H = (M^{-1}N)^{\mathrm{T}}A(M^{-1}N) - (M^{-1}N)^{\mathrm{T}}(P^{-1}Q)^{\mathrm{T}}A(P^{-1}Q)(M^{-1}N),$$

and therefore $R + H = A - T^{\mathrm{T}}AT$ is positive definite. The induced splitting is given by (11). This splitting is P-regular, as is clear from the identity

$$A - T^{\mathrm{T}}AT = A - (B^{-1}C)^{\mathrm{T}}A(B^{-1}C) = (B^{-1}A)^{\mathrm{T}}(B + B^{\mathrm{T}} - A)(B^{-1}A)$$

and the first part of the proof. □

Remark 3.6. In the important special case $M = P^{\mathrm{T}}$, B is actually SPD, since

$$B^{-1} = M^{-\mathrm{T}}(M + M^{\mathrm{T}} - A)M^{-1}.$$

It follows that T is symmetrizable, and therefore Chebyshev or CG acceleration can be used. Besides SSOR, an important example is provided by alternating iterations based on a splitting of the form $A = A_1 + A_2$ with A_1, A_2 positive definite and such that $A_2 = A_1^{\mathrm{T}}$; see, e.g., [14]. Letting $M = rI + A_1$, $N = rI - A_2$, $P = rI + A_2$ and $Q = rI - A_1$, with $r > 0$, the splittings $A = M - N = P - Q$ are P-regular (because $M + M^{\mathrm{T}} - A = 2rI$), and $M = P^{\mathrm{T}}$. The convergence of the corresponding alternating iteration is insured by Theorem 3.5.

Remark 3.7. The unique splitting $A = B - C$ such that $B^{-1}C = T = P^{-1}QM^{-1}N$ may be P-regular, and hence convergent, even if the splittings $A = M - N = P - Q$ are not P-regular. This shows the usefulness of rewriting the alternating iteration as a single splitting. An example is provided by the Alternating Direction Implicit methods (ADI), in which the SPD matrix A is split as $A = A_1 + A_2$ with A_1 and A_2 symmetric positive definite. If M, N, P and Q are defined as in Remark 3.6, with $r > 0$, it is easy to see with examples that the splittings $A = M - N = P - Q$ are not P-regular, generally speaking. Nevertheless, it is possible to associate a single splitting $A = B - C$ to the iteration, with

$$B^{-1} = P^{-1}(M + P - A)M^{-1} = (rI + A_2)^{-1}(2rI)(rI + A_1)^{-1}.$$

It follows that B is invertible for all $r > 0$, with $B = \frac{1}{2r}(rI + A_1)(rI + A_2)$, and therefore $B + B^{\mathrm{T}} - A = rI + \frac{1}{2r}(A_1A_2 + A_2A_1)$. In the commutative case (i.e., when $A_1A_2 = A_2A_1$) we find that $B + B^{\mathrm{T}} - A$ is SPD because A_1A_2, as the product of two SPD matrices, has real positive eigenvalues. Hence, $A = B - C$ is a P-regular splitting (for all $r > 0$) and therefore $\rho(T) < 1$. Moreover, T is symmetrizable and acceleration techniques can be used. However, if the commutativity condition is not satisfied, it may happen that $A = B - C$ is a P-regular splitting only for sufficiently large r, yet the ADI iteration converges for all $r > 0$.

We next analyze the case where A is symmetric positive semidefinite, using as a tool the following generalization of Stein's Theorem to the singular case.

Lemma 3.8. [8] *A matrix T is convergent if and only if there exist two symmetric positive semidefinite matrices Z, Y, such that $Z = Y - T^{\mathrm{T}}YT$ and $\mathcal{N}(I - T) = \mathcal{N}(Y) = \mathcal{N}(Z)$.*

Theorem 3.9. *Let A be symmetric positive semidefinite and singular. If the splittings $A = M - N = P - Q$ are P-regular, then $T = P^{-1}QM^{-1}N$ is convergent. Moreover, T induces infinitely many splittings of A, and they are all P-regular.*

Proof. First we prove that $M + P - A$ is nonsingular, showing that the compatibility condition (3) is verified. Since $A = M - N = P - Q$ are P-regular splittings, matrices $M + M^{\mathrm{T}} - A$ and $P + P^{\mathrm{T}} - A$ are both positive definite. But since the symmetric part of $M + P - A$ is

$$(M+P-A)^{\mathrm{S}} = (M+P-A)/2+(M^{\mathrm{T}}+P^{\mathrm{T}}-A)/2 = (M+M^{\mathrm{T}}-A)/2+(P+P^{\mathrm{T}}-A)/2,$$

$M + P - A$ is positive definite and therefore nonsingular. Hence $B^{-1} = P^{-1}(M + P - A)M^{-1}$ is well-defined and $B^{-1}C = T$. To prove that T is convergent, we apply Lemma 3.8 with $Y = A$, the original coefficient matrix. The same argument used in the nonsingular case (Theorem 3.5) shows that $Z = A - T^{\mathrm{T}}AT$ is now positive semidefinite. Also, $I - T = B^{-1}A$ implies $\mathcal{N}(I - T) = \mathcal{N}(A)$. We only have left to show that $\mathcal{N}(Z) = \mathcal{N}(A)$. First we show that $\mathcal{N}(A) \subseteq \mathcal{N}(Z)$. Let $v \in \mathcal{N}(A)$, then $v = Tv$ and therefore $Zv = Av - T^{\mathrm{T}}ATv =$

$Av - T^{\mathrm{T}}Av = 0$, hence $v \in \mathcal{N}(Z)$. To prove the inclusion in the other direction, let $R = A - (M^{-1}N)^{\mathrm{T}}A(M^{-1}N)$, $S = A - (P^{-1}Q)^{\mathrm{T}}A(P^{-1}Q)$. Since $M^{-1}N$ is convergent, using Lemma 3.8, we have that $\mathcal{N}(R) = \mathcal{N}(A)$. Since $Z = R + (M^{-1}N)^{\mathrm{T}}S(M^{-1}N)$ is a symmetric positive semidefinite matrix, it follows that $Zv = 0$ if and only if $v^{\mathrm{T}}Zv = 0$ (see [11, p. 400]), therefore $v \in \mathcal{N}(Z)$ if and only if $v^{\mathrm{T}}Rv = -v^{\mathrm{T}}(M^{-1}N)^{\mathrm{T}}S(M^{-1}N)v$. In this equality, the quantity on the left-hand side is nonnegative, the one on the right-hand side is nonpositive and therefore they must be both zero. Hence, $Rv = 0$, but since $\mathcal{N}(R) = \mathcal{N}(A)$, $v \in \mathcal{N}(A)$ and the convergence of T is established.

Finally, we show that the splitting $A = B - C$ is P-regular. We have just shown that $Zv = 0$ if and only if $Av = 0$; because Z is symmetric positive semidefinite, this is equivalent to saying that $v^{\mathrm{T}}Zv = 0$ if and only if $Av = 0$, or that $(B^{-1}Av)^{\mathrm{T}}(B + B^{\mathrm{T}} - A)(B^{-1}Av) = 0$ if and only if $Av = 0$. Clearly, this is equivalent to saying that $B + B^{\mathrm{T}} - A$ is symmetric positive definite, that is, $A = B - C$ is a P-regular splitting. This argument is valid for *any* of the infinitely many splittings $A = B - C$ such that $B^{-1}C = P^{-1}QM^{-1}N$, and not just for the splitting (11). □

Many of the results of this section and the next, can be extended to alternating schemes involving more than two splittings of the coefficient matrix A. For example, in the solution of three-dimensional problems it is useful to consider three splittings $A = M - N = P - Q = R - S$ and the corresponding three-step alternating procedure. This requires studying the convergence of the iteration matrix $T = R^{-1}SP^{-1}QM^{-1}N$. The extension to this case, or to an arbitrary number of splittings, can be done by recursively applying the results shown for two splittings.

4. A comparison theorem

We have already observed that the convergence of the two splittings $A = M - N = P - Q$ is not sufficient to insure the convergence of the alternating iteration (8). Even if the alternating iteration converges, in general there is no guarantee that it will converge faster than either of the two basic splittings. However, the following result shows that under appropriate conditions the asymptotic rate of convergence of the alternating iteration (8) is at least as good as the rate of convergence of the fastest of the two basic iterations.

Theorem 4.1. *Let A be a monotone matrix. If the splittings $A = M - N = P - Q$ are regular, the following upper bound on the spectral radius of $T = P^{-1}QM^{-1}N$ holds*

$$\rho(T) \leq \min(\rho(M^{-1}N), \rho(P^{-1}Q)). \tag{12}$$

Proof. Let T be the iteration matrix corresponding to the induced splitting (11). We know from Theorem 3.2 that $A = B - C$ is a weak regular splitting. We have the following two matrix inequalities

$$B^{-1} = M^{-1} + P^{-1}NM^{-1} \geq M^{-1}, \tag{13}$$
$$B^{-1} = P^{-1} + P^{-1}QM^{-1} \geq P^{-1}. \tag{14}$$

We can now apply the comparison theorem for weak regular splittings due to Elsner [7] to $A = B - C$ and $A = M - N$ to get $\rho(T) \leq \rho(M^{-1}N)$. Applying the same result to $A = B - C$ and $A = P - Q$ we also get $\rho(T) \leq \rho(P^{-1}Q)$. Therefore, the upper bound (12) on $\rho(T)$ holds. □

As was shown in [6] and in [22], considerable savings in arithmetic operations are possible when implementing alternating iterations of the type (8). Together with our comparison result, this shows that alternating between two splittings can be advantageous over iterating with a single splitting, if A is monotone. Nevertheless, it would be desirable to have comparison theorems of the type (12) with strict inequality. This can be achieved by requiring that the inequalities (13) and (14) be strict; see [15]. To that end one can require that $P^{-1} > 0$ and that neither N or Q has any zero column. Furthermore, the assumptions in Theorem 4.1 can be somewhat weakened but in that case the hypotheses may be hard to check; see [15].

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