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Convergence of nested classical iterative methods for linear systems

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Dedicated to Richard S. Varga on the occasion of his sixtieth birthday

Summary. Classical iterative methods for the solution of algebraic linear systems of equations proceed by solving at each step a simpler system of equations. When this system is itself solved by an (inner) iterative method, the global method is called a two-stage iterative method. If this process is repeated, then the resulting method is called a nested iterative method. We study the convergence of such methods and present conditions on the splittings corresponding to the iterative methods to guarantee convergence for *any* number of inner iterations. We also show that under the conditions presented, the spectral radii of the global iteration matrices decrease when the number of inner iterations increases. The proof uses a new comparison theorem for weak regular splittings. We extend our results to larger classes of iterative methods, which include iterative block Gauss-Seidel. We develop a theory for the concatenation of such iterative methods. This concatenation appears when different numbers of inner iterations are performed at each outer step. We also analyze block methods, where different numbers of inner iterations are performed for different diagonal blocks.

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1 Introduction

We consider here certain iterative methods for the solution of the algebraic linear system of equations

$$Ax=b, \tag{1}$$

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*** *Present address:* Department of Mathematics, Temple University, Philadelphia, PA 19122, USA where x and b are vectors, and A is a square nonsingular matrix. It is customary to view classical iterative methods as the repeated solution of

$$M x_{k+1} = b + N x_k \tag{2}$$

where $A = M - N$ is called a splitting of the matrix A, M is nonsingular, and x_0 is given. Varga [25, 26] pioneered the study of such methods; see also Young [29] and Ortega and Rheinboldt [21]. These methods are especially important in building algorithms for parallel computers; see [2, 19], and [23].

The method (2) is a natural formulation for systems arising from discretizations of differential equations and it is generally assumed that the system

$$M v = g \tag{3}$$

can be solved with considerably less computional effort than (1). Here we consider the iterative solution of the system (3), called the *inner* iteration, at each iteration of (2), the *outer* iteration. Again, the customary way of looking at the inner iteration is by the splitting $M = B - C$ and the repeated solution of

$$B v_{j+1} = g + C v_j. \tag{4}$$

These are often called two-stage iterative methods [11, 18]. There is a wide range of application for these methods, and, in particular, our theory applies to block iterative methods [26]. In fact, our theory applies to iterative methods which can be described by an error propagation equation of the form $e_{k+1} = T e_k$, where e_k is the error at step k of the iteration and T, the iteration matrix, is convergent; see Sect. 2.

Two-stage methods, also called inner/outer methods, have been applied to fictitious components and to domain decomposition methods; see [9, 24] and the references given therein. Golub and Overton [11, 12] have considered two-stage methods when the outer iteration is the Chebyshev or the Richardson method. For nonlinear systems of equations, methods with outer nonlinear iteration and linear inner iteration have been extensively studied and have been applied to different areas of science and engineering; see e.g., Bank and Rose [1], Dembo et al. [6] or Díaz et al. [7].

Nichols [18] studied two-stage methods for the solution of (1) with general inner iteration methods. She showed that if the outer and the inner iterations are convergent then, for a large enough number of inner iterations, p, the two-stage method is convergent; see also Wachspress [27]. In Lemma 2.3 we note that a very general class of iterative methods can be represented by corresponding (unique) splittings and thus, without loss of generality, both the outer and the inner iterations are represented by splittings. In Theorem 4.2 we set conditions on the splittings so that convergence is guaranteed for *any* number of inner iterations. Moreover, under these conditions, we show that the spectral radii of the global iteration matrices decrease when the number of inner iterations increases. This last intuitive result is an initial step for a strategy to find an "optimal" number of inner iterations, but may not hold if the conditions of our theorem are violated; see Sect. 5.

The conditions we set relate to regular and weak regular splittings. These arise naturally in many applications and have been widely studied [3, 20, 26, 29]. Our proofs are based on the theory of nonnegative matrices, the Perron-

Frobenius theorem and also on comparison theorems; see the mentioned references and also [5]. In Sect. 3 we strengthen a comparison theorem for weak regular splittings which we use to develop our theory of convergence for the iterative methods studied here.

We note here that, except for the trivial case of one inner iteration, a two-stage method of the form (2–4) does not adequately describe the Block Gauss-Seidel method [3, 26, 29] when the diagonal block equations themselves are solved iteratively. This fact is often overlooked in the literature; see, e.g., Rodrigue [24]. Consider $A=D-L-U$, where D is block diagonal, L is strictly block lower triangular and U is strictly block upper triangular. For the purpose of illustration, consider the matrix A partitioned into $q\times q$ blocks, i.e.,

$$Ax=\begin{bmatrix} D_1 & -U_{12} & \dots & -U_{1q} \\ -L_{21} & D_2 & \dots & -U_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -L_{q1} & -L_{q2} & \dots & D_q \end{bmatrix}\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \dots \\ x^{(q)} \end{bmatrix}=\begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \dots \\ b^{(q)} \end{bmatrix}.$$

The block Gauss-Seidel method is

for $k=0,1,\dots$
  for $i=0,1,\dots,q$
    $D_i x_{k+1}^{(i)}=b^{(i)}+\sum_{j<i} L_{ij} x_{k+1}^{(j)}+\sum_{j>i} U_{ij} x_k^{(j)}$

which can be represented as

$$(D-L)\,x_{k+1}=b+U\,x_k, \tag{5}$$

where it is understood that the solution at each step proceeds one block at a time. Comparing (5) with (2) and applying the philosophy of two stage methods, one would have to solve, at each step of (5), a linear system by splitting $(D-L)=V-W$, say. This is not how blocks methods are solved and, in addition, it would be an expensive iteration. The usual approach, instead, is to use an iterative method for the solution of the systems corresponding to each diagonal block D_i. To study the usual block Gauss-Seidel method, we write (5) as

$$D x_{k+1}=b+U x_k+L x_{k+1}, \tag{6}$$

and one can think of it as an implicit method, where at each step we solve a system of the form

$$D x=r(x_k, x_{k+1}, b), \tag{7}$$

in which r is a function of the known iterate x_k, the iterate to be determined x_{k+1}, and b, a given vector. Of course, due to the block triangular structure of $D-L$, the system (7) is not truly implicit, since the needed components of x_{k+1} are always available. The solution of (7) by an iterative method can be represented by a splitting of the block diagonal matrix $D=F-G$, i.e., each diagonal block has a splitting $D_i=F_i-G_i$. This produces the following

Algorithm 1.1 (Iterative Block Gauss Seidel).

for $k=0, 1, \ldots$
  $y_0=y=(y^{(1)}, y^{(2)}, \ldots, y^{(q)})^T=x_k$
  for $i=1$ **to** q
    for $j=0$ **to** $p_{ik}-1$
      $F_i\, y_{j+1}^{(i)}=(b+U x_k+L x_{k+1})^{(i)}+G_i\, y_j^{(i)}$
    $x_{k+1}^{(i)}=y_{p_{ik}}^{(i)}$

The notion of *composite splittings* enables us to study methods such as block Gauss-Seidel, or SOR (with $\omega<1$). In Sect. 4 we study nested iterative methods where, e.g., the solution of the system (4) is itself solved by another iteration, and this recursive idea is repeated for a certain number of levels of nesting. For these methods and for iterative block Gauss-Seidel our global convergence results also apply; see Sect. 4. Furthermore, the monotonicity result also holds; namely, under certain conditions, if two nested methods have inner iteration matrices whose spectral radii compare in one direction, say $\rho(R_1)\leqq\rho(R_2)$, then the global iteration matrices compare in the same direction; see Sect. 5.

In dynamic nested iterative methods the number of inner iterations may change at each outer iteration. This amounts to concatenating different iterative methods and, since the product of two matrices with spectral radius less than unity may have spectral radius greater than one, the resulting method might not be convergent. In Sect. 6 we address this problem and show that, under a slightly more restrictive hypothesis, dynamic nested iterative methods are convergent. Finally, in Sect. 7 we study convergence and monotonicity results of block methods.

The theory presented herein is a first step toward the development of parallel block chaotic relaxation methods [2, 4, 15, 23] in which the linear systems corresponding to diagonal blocks are solved by iterative methods.

2 Preliminaries

In this section, we give some notation, present some basic results and review some definitions; see further [3, 20–22, 26], and [29]. By $\rho(B)$ we denote the spectral radius of the matrix B. We say that a matrix B is *convergent* if $\rho(B)<1$. We say that a vector x is nonnegative, denoted $x\geqq 0$, if all its entries are nonnegative. Define $x>0$ as $x\geqq 0$ with each component $x_i\neq 0$ for all i. Similarly, a matrix B is said to be nonnegative, denoted $B\geqq 0$, if all its entries are nonnegative or, equivalently, if it leaves invariant the set of all vectors with nonnegative entries. We compare two matrices $A\geqq B$, when $A-B\geqq 0$. By I_m we denote the $m\times m$ identity matrix and when the order of the identity matrix is clear from the context, we simply denote it by I.

We define $A=M-N$ as a *splitting* of A when M is nonsingular. We say that the splitting is *convergent* if $\rho(M^{-1}N)<1$; *regular* if $M^{-1}\geqq 0$ and $N\geqq 0$; and *weak regular* if $M^{-1}\geqq 0$, $M^{-1}N\geqq 0$ and $NM^{-1}\geqq 0$. Obviously, a regular splitting is a weak regular splitting but the converse is not always true. We define $A=M-N_1-N_2$ as a *composite splitting* of A when M is nonsingular.

We say that $A=M-N_1-N_2$ is a *convergent regular composite splitting* if both $M_1:=M-N_2$ and $A=M_1-N_1$ are convergent regular splittings.

In the following lemma we collect several results on splittings and nonnegative matrices. The proofs can be found, e.g., in [3] and [22].

Lemma 2.1. *Let $A=M-N$ be a weak regular splitting. Let $T\geqq 0$. Then the following hold:*

(a) *A is nonsingular and $A^{-1}\geqq 0$ if and only if $\rho(M^{-1}N)<1$.*
(b) *$\rho(T)<1$ if and only if $(I-T)^{-1}$ exists and $(I-T)^{-1}\geqq 0$.*
(c) *If there exists $z\geqq 0$, $z\neq 0$ and a scalar $\alpha>0$ such that $\alpha z\leqq Tz$, then $\alpha\leqq\rho(T)$.*
(d) *If there exists $x>0$ and a scalar $\alpha>0$ such that $Tx\leqq\alpha x$, then $\rho(T)\leqq\alpha$.*

The following example shows that the condition of strict positivity of the vector x is essential.

Example 2.2.

$$T=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix},\quad x=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \rho(T)=1 \quad\text{and}\quad Tx\leqq\alpha x, \quad \text{for all } \alpha>0.$$

The following result, although straightforward, plays an important role in our analysis.

Lemma 2.3. *Given a nonsingular matrix A and T such that $(I-T)^{-1}$ exists, there exists a unique pair of matrices M, N, such that $T=M^{-1}N$ and $A=M-N$, where M nonsingular.*

Proof. Consider $M=A(I-T)^{-1}$ and $N=M-A$. Then $M^{-1}N=M^{-1}(M-A)=I-(I-T)A^{-1}A=T$. For the uniqueness, let $A=\tilde{M}-\tilde{N}$ be a splitting of A such that $T=\tilde{M}^{-1}\tilde{N}$. Then $\tilde{M}T=\tilde{M}-A$ and thus $\tilde{M}=A(I-T)^{-1}=M$. □

In the context of Lemma 2.3 we say that T *induces* the unique splitting $A=M-N$.

3 Comparison theorem

In this section we present a comparison theorem between weak regular splittings of the same matrix. It strengthens some comparison theorems in Varga [26], Csordas and Varga [5] and Elsner [8]; see also Miller and Neumann [16].

Theorem 3.1 (Comparison Theorem). *Let $A=M-N=\tilde{M}-\tilde{N}$ be convergent weak regular splittings such that*

$$\tilde{M}^{-1}\geqq M^{-1}, \tag{8}$$

and let x and z be the nonnegative Frobenius eigenvectors of $T=M^{-1}N$ and $\tilde{T}=\tilde{M}^{-1}\tilde{N}$, respectively. If $\tilde{N}z\geqq 0$ or if $Nx\geqq 0$ with $x>0$, then

$$\rho(\tilde{T})\leqq\rho(T). \tag{9}$$

Proof. If $\rho(\tilde{T})=0$ then the theorem is trivially true; we shall therefore only consider $\rho(\tilde{T})\neq 0$. Assume that $\tilde{N}z\geqq 0$. We have that $\tilde{T}z=\tilde{M}^{-1}\tilde{N}z=\rho(\tilde{T})z$, which implies that

$$\tilde{M}z=\frac{1}{\rho(\tilde{T})}\tilde{N}z.$$

Hence

$$Az=\tilde{M}(I-\tilde{T})z=\frac{1-\rho(\tilde{T})}{\rho(\tilde{T})}\tilde{N}z\geqq 0.$$

From (8) it follows that

$$(I-\tilde{T})z=\tilde{M}^{-1}Az\geqq M^{-1}Az=(I-T)z,$$

therefore $\tilde{T}z=\rho(\tilde{T})z\leqq Tz$ and by Lemma 2.1(c), $\rho(\tilde{T})\leqq\rho(T)$. The proof for the case $Nx\geqq 0$ with $x>0$ is analogous. □

The proof of Theorem 3.1 is similar to that in [17, Lemma 2.2], where the comparison is done between splittings of two different matrices A_1 and A_2; see also the recent paper by Marek and Szyld [14].

Varga [26] showed that if $A=M-N=\tilde{M}-\tilde{N}$ are regular splittings,

$$\tilde{N}\leqq N \tag{10}$$

implies the result (9). Csordas and Varga [5] and Woźnicki [28] proved the same result with the weaker hypothesis (8). Some still weaker conditions were set in [5, 16] always requiring the splittings to be regular. Elsner [8] proved the following

Lemma 3.2. *Let $A^{-1}\geqq 0$ and let $A=M-N=\tilde{M}-\tilde{N}$ be weak regular splittings. If* (10) *holds, then $\rho(\tilde{M}^{-1}\tilde{N})\leqq\rho(M^{-1}N)$.*

He also showed that (8) with either $N\geqq 0$ or $\tilde{N}\geqq 0$ imply (9). Here we have shown that even if the matrices N or $\tilde{N}$ do not map the entire set of nonnegative vectors into itself, the result (9) holds if N or $\tilde{N}$ map particular nonnegative vectors into that set. Moreover, all splittings we consider in this paper satisfy the conditions of Theorem 3.1, namely that if v is the Frobenius eigenvalue of $T=M^{-1}N$, then $Nv\geqq 0$; see Theorem 5.1.

4 Convergence of two-stage and nested iterations

The most general method of solution of (1) considered in this paper is nested block iterations. Iterative block Gauss-Seidel (Algorithm 1.1) is a special case of the following algorithm, where $A=M-N_1-N_2$ is partitioned into $q\times q$ blocks, and $M=F-G$ is block diagonal.

Algorithm 4.1 (Nested Block Iteration).

> **for** $k=0, 1, \ldots$
> $y_0=y=(y^{(1)}, y^{(2)}, \ldots, y^{(q)})^T=x_k$
> **for** $i=1$ **to** q
> **for** $j=0$ **to** $p_{ik}-1$

$$f_i\, y_{j+1}^{(i)} = (b + N_1\, x_k + N_2\, x_{k+1})^{(i)} + G_i\, y_j^{(i)}$$
(11) $$x_{k+1}^{(i)} = y_{p_{ik}}^{(i)}.$$

In most practical applications the matrix N_2 is lower block triangular. In this case, the step (11) is explicit, since the needed components of x_{k+1} are all known. For general composite splittings $A = M - N_1 - N_2$, the step (11) is truly implicit and usually does not appear in practice. For completeness, we present our results for general matrices N_1 and N_2, since all convergence theorems apply to this general case.

In this section we first consider *Two-stage Methods*, i.e. the case $q=1$ and $p_{1k} = p$ for all k. The convergence results for the two-stage method are used later to analyze nested iterative methods and in Sect. 7 to study the general case. We first compute the global iteration matrix and then study the convergence properties of this method. We begin by replacing the loop over j (Eq. (11)) with the equation

$$\begin{aligned} y_p &= \sum_{i=0}^{p-1} (F^{-1} G)^i F^{-1} (b + N_1\, x_k + N_2\, x_{k+1}) + (F^{-1} G)^p\, y_0 \\ &= (I - H^p)(I - H)^{-1} F^{-1} (b + N_1\, x_k + N_2\, x_{k+1}) + H^p\, y_0, \end{aligned}$$

where $H = F^{-1} G$ is the iteration matrix of one step of the inner iteration and we have used the identity $(I - H^p)(I - H)^{-1} = \sum_{i=0}^{p-1} (F^{-1} G)^i$. We may therefore rewrite (11) with the equation:

$$F(I - H)(I - H^p)^{-1}\, y_p = b + N_1\, x_k + N_2\, x_{k+1} + F(I - H)(I - H^p)^{-1} H^p\, y_0.$$

Let $B := F(I - H)(I - H^p)^{-1}$ and $C := F(I - H)(I - H^p)^{-1} H^p$. We represent (11) as

$$B\, y_p = b + N_1\, x_k + N_2\, x_{k+1} + C\, y_0; \quad \text{i.e.,}$$
(12) $$B\, x_{k+1} = b + N_1\, x_k + N_2\, x_{k+1} + C\, x_k.$$

In the context of Lemma 2.3, B and C are the unique matrices *induced* by the iteration matrix $R = H^p = B^{-1} C$ on the matrix $M = (F - G) = B - C$. The iterative method (12) corresponding to the splitting $M = B - C$ is not generally used in actual computations. This is a convenient device to study two-stage methods. Furthermore, the fact that we consider an inner iteration with iteration matrix of the specific form $R = (F^{-1} G)^p$ is not used in our analysis. In the remainder of the paper we often use the concept of a splitting induced by a given iteration matrix R.

Although the method (12) is implicit, the iteration matrix is clearly

(13) $$T = (B - N_2)^{-1} (C + N_1)$$

and the (unique) matrices induced by T on A are

(14) $$M_T := B - N_2 = M(I - R)^{-1} - N_2$$

(15) $$N_T := C + N_1 = M(I - R)^{-1} R + N_1.$$

The case $N_2=0$ was studied by Nichols [18], and (13) becomes

$$T=B^{-1}N_1+R=R+S-RS=I-(I-R)(I-S), \tag{16}$$

where $S=M^{-1}N_1$, $M=B-C$ and $R=B^{-1}C$. It should be noted, however, that the two-stage method studied here encompasses a larger class of methods. In particular, iterative block Gauss-Seidel cannot be represented by an iteration matrix of the form (16).

Theorem 4.2. *Let $A=M-N_1-N_2$ be a convergent regular composite splitting and let $R\geqq 0$, $\rho(R)<1$. If the (unique) splitting $M=B-C$ such that $R=B^{-1}C$ is weak regular, then the iterative method defined by*

$$A=M_T-N_T, \quad M_T:=M(I-R)^{-1}-N_2, \quad \text{and} \quad N_T:=M(I-R)^{-1}R+N_1, \tag{17}$$

is convergent. Moreover, $A=M_T-N_T$ is a weak regular splitting.

Since the use of the induced splitting $M=B-C$ is a technical device, conditions on it are not expected to be directly verified. We see later that certain iterative methods, including the two-stage method, induce splittings satisfying the hypothesis of the theorem.

Proof of Theorem 4.2. Since $B^{-1}\geqq 0$, $N_2\geqq 0$ and $M^{-1}N_2$ is convergent, the inequality $M^{-1}N_2=(I-R)^{-1}B^{-1}N_2\geqq B^{-1}N_2\geqq 0$ implies that $B^{-1}N_2$ is a convergent nonnegative matrix. Therefore, since $B^{-1}\geqq 0$ and $N_1\geqq 0$,

$$T=(B-N_2)^{-1}(C+N_1)=(I-B^{-1}N_2)^{-1}(R+B^{-1}N_1)\geqq 0.$$

Let $S=(M-N_2)^{-1}N_1$, then

$$\begin{aligned}(I-T)^{-1}&=A^{-1}M_T=(M-N_1-N_2)^{-1}(M(I-R)^{-1}-N_2)\\&=(I-S)^{-1}(M-N_2)^{-1}M((I-R)^{-1}-M^{-1}N_2)\\&=(I-S)^{-1}(I-M^{-1}N_2)^{-1}((I-R)^{-1}R+I-M^{-1}N_2)\\&=(I-S)^{-1}((I-M^{-1}N_2)^{-1}(I-R)^{-1}R+I)\geqq 0.\end{aligned}$$

Therefore, by Lemma 2.1(b), T is convergent. Furthermore,

$$M_T^{-1}=(I-B^{-1}N_2)^{-1}B^{-1}\geqq 0.$$

Since $\rho(N_2B^{-1})=\rho(B^{-1}N_2)<1$, and since $CB^{-1}\geqq 0$,

$$\begin{aligned}N_TM_T^{-1}&=(M(I-R)^{-1}R+N_1)(I-(I-R)M^{-1}N_2)^{-1}(I-R)M^{-1}\\&=M(I-R)^{-1}R(I-B^{-1}N_2)^{-1}B^{-1}+N_1(I-B^{-1}N_2)^{-1}B^{-1}\\&=C(I-B^{-1}N_2)^{-1}B^{-1}+N_1(I-B^{-1}N_2)^{-1}B^{-1}\\&=CB^{-1}(I-N_2B^{-1})^{-1}+N_1(I-B^{-1}N_2)^{-1}B^{-1}\geqq 0. \quad \square\end{aligned}$$

Note that the splitting $A=M_T-N_T$ is not necessarily regular since N_T may have some of the negative entries that may have been in C. In the remainder of this section we apply Theorem 4.2 to practical iterative methods. In the following corollary we present conditions for convergence of two-stage methods *independent* of the number of inner iterations, p.

Corollary 4.3. *Let $A=M-N_1-N_2$ be a convergent regular composite splitting and let p be a nonnegative integer. If $M=F-G$ is a weak regular splitting, then the two-stage iterative method is convergent, and its induced splitting is weak regular.*

Proof. To apply Theorem 4.2 it suffices to show that $M=B-C$ is a weak regular splitting with $B=M(I-(F^{-1}G)^p)^{-1}$ and $C=B-M$. First note that

$$\begin{aligned} B^{-1}&=(I-(F^{-1}G)^p)\,M^{-1}\\ &=(I-(F^{-1}G)^p)(I-(F^{-1}C))^{-1}F^{-1}=\sum_{i=0}^{p-1}(F^{-1}G)^iF^{-1}\geqq 0.\end{aligned}$$

Since $M=F-G$ is a weak regular splitting $F^{-1}C\geqq 0$ and $CF^{-1}\geqq 0$. Thus $B^{-1}C=(F^{-1}G)^p\geqq 0$, and

$$\begin{aligned} CB^{-1}&=M(I-(F^{-1}G)^p)^{-1}(F^{-1}C)^p(I-(F^{-1}C)^p)\,M^{-1}\\ &=M(F^{-1}G)^pM^{-1}=(CF^{-1})^p\geqq 0. \quad \square\end{aligned}$$

The following example shows that the hypotheses of Corollary 4.3 (and by extension those of Theorem 4.2) can not be weakened. Let T_p be the iteration matrix for a two-stage method with p inner iterations.

Example 4.4.

$$A=\begin{bmatrix}1.25 & -1 & 0.25\\ -1 & 1.5 & -1\\ 0.25 & -1 & 1.25\end{bmatrix},\quad M=\begin{bmatrix}1 & -0.5 & 0\\ -0.5 & 1 & -0.5\\ 0 & -0.5 & 1\end{bmatrix},\quad F=I,\quad N_2=0,$$

$$M^{-1}N_1=R=G=\begin{bmatrix}0 & 0.5 & 0\\ 0.5 & 0 & 0.5\\ 0 & 0.5 & 0\end{bmatrix},\quad T_1=F^{-1}N_1+R=\begin{bmatrix}-0.25 & 1 & -0.25\\ 1 & -0.5 & 1\\ -0.25 & 1 & -0.25\end{bmatrix}.$$

Thus, $A=M-N_1$ is a weak regular splitting and $M=F-G$ is a regular splitting. In this example $\rho(M^{-1}N_1)=\rho(R)=0.71$ and thus the inner and outer iterations are both convergent; but $\rho(T_1)=1.9$ and the two-stage iteration with 1 inner iteration is not convergent. Thus in this example a convergent iteration within a convergent iteration is divergent.

Rodrigue [24, Theorem 4.1] showed that when $N_2=0$, $p=1$ and both the outer and the inner iterations correspond to regular splittings, then the overall method is convergent and, in particular, induces a regular splitting. For the case $p>1$ and the same hypothesis, as the following example shows, the global induced splitting may not be regular.

Example 4.5.

$$A=M=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix},\quad F=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\quad ,G=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$$H=F^{-1}G=\begin{bmatrix}0 & 0.5\\ 0.5 & 0\end{bmatrix},\quad N_{T_2}=M(I-H^2)^{-1}H^2=\begin{bmatrix}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{bmatrix}.$$

The fact that N_{T_2} has negative entries also implies that the conditions in the comparison theorem by Elsner [8] are not satisfied either; see Sect. 3.

The theory we are developing can be extended to the case of recursive inner iterations. We call these *Nested Iterative Methods.* Consider the solution of the system $Ax=b$ by a two-stage method with the outer iteration defined by the composite splitting $A=M-N_1-N_2$. Then, instead of solving the system $Mx=r(x_k, x_{k+1}, b)$ by an iterative method, it is solved by a two-stage method, and so on. This implies the specification of a new composite splitting, with iteration matrix R, $R=F^{-1}G$, $M=F-G$, and the number of iterations (p) at each level. One could interpret $M=F-G$ as (14–15). Of course, at the last (innermost) level, the system is solved by an iterative method, say with $M=B-C$. After the formal recursive definition below, we show that under conditions similar to those in Corollary 4.3, nested iterations are convergent *independent* of the number of iterations at each level.

Definition 4.6. Consider the system $Ax=b$. Let $A=M-N_1-N_2$ be a composite splitting and p be a nonnegtive integer. In a Nested Iteration the equation $Mx=b+N_1 x_k+N_2 x_{k+1}$ is solved by either,

1. p steps of an iterative method (note that the effect of this is to solve $Ax=b$ by a two-stage iteration),

or,

2. p iterations of the Nested Iteration with iteration matrix R (replace A by M and b by $(b+N_1 x_k+N_2 x_{k+1})$ and apply the definition again).

Corollary 4.7. *At each level let the redefined $A=M-N_1-N_2$ (i.e. in part 2 of the definition) be a convergent regular composite splitting and at the innermost level let $M=B-C$ be a weak regular splitting (i.e., in part 1 of the definition). Then the corresponding nested iterative method is convergent.*

Proof. By induction on the number of levels of nesting. The method with one level of nesting is the two-stage method, and Corollary 4.3 applies. For the inductive step, assume that the induced splitting of R on $M=B-C$ is weak regular. Then, using arguments similar to those in Corollary 4.3, $\hat{F}=M(I-R^p)^{-1}$ and $\hat{G}=M(I-R^p)^{-1}R^p$ from a weak regular splitting. Therefore, by Theorem 4.2 the method is convergent and if T is the iteration matrix, $A=M_T-N_T$ is a weak regular splitting, where $M_T=A(I-T)^{-1}$. □

Consider the particular case of nested iteration, where only one step of the first inner iteration is performed. We show in the following theorem that such a method can be represented by a two-stage method corresponding to a particular composite splitting. Moreover, the result can be extended, by a simple induction argument, to show that if there are multiple levels of nested iteration with one inner iteration at every level (except the last) then the whole iteration reduces to a two-stage method. This is a natural situation, and the theorem shows that no matter how many levels of one iteration are used, the nested method can be viewed simply as a two-stage method.

Theorem 4.8. *Consider the solution of $Ax=b$ by a nested iteration with $A=M-N_1-N_2$. Assume that the (outer) system $Mx=b+N_1 x_k+N_2 x_{k+1}$ is solved by only one step of the (inner) two-stage iteration corresponding to the composite splitting $M=\hat{M}-\hat{N}_1-\hat{N}_2$. Assume further that the corresponding (inner) system* *is solved with p iterations of the method corresponding to the splitting $\hat{M}=F-G$. Then the resulting iterative method is given by the composite splitting $A=M-\tilde{N}_1-\tilde{N}_2$ where $\tilde{N}_1=N_1+\hat{N}_1$ and $\tilde{N}_2=N_2+\hat{N}_2$.*

Proof. Let $R=F^{-1}G$. From Eq. (13) we have that the inner iteration matrix is $T_1=(F_1-\hat{N}_2)^{-1}(G_1+\hat{N}_1)$, where $F_1^{-1}G_1=R^p=(F^{-1}G)^p$. Thus $M_{T_1}=F_1-\hat{N}_2$ and $N_{T_1}=G_1+\hat{N}_1$. The iteration matrix for the global (outer) iteration is $T_2=(F_2-N_2)^{-1}(G_2+N_1)$, where $M=F_2-G_2$ is the splitting corresponding to the inner iteration, and $F_2=M_{T_1}$ and $G_2=N_{T_1}$. Therefore $T_2=(F_1-\hat{N}_2-N_2)^{-1}(G_1+\hat{N}_1+N_1)$. The theorem follows by substituting $\tilde{N}_1=N_1+\hat{N}_1$ and $\tilde{N}_2=\hat{N}_2+N_2$. □

5 Monotonicity

In the previous section we gave sufficient conditions for the convergence of nested iterative methods. In this section we show that, under the same conditions, if the spectral radius of the inner iteration matrix decreases (e.g., increasing the number of inner iterations) then, so does the spectral radius of the global iteration matrix. For example, in the case of two-stage methods, the spectral radius of the global iteration matrix, $\rho(T_p)$, is a monotonically decreasing function of p. This result is intuitive but, as we see later, if the conditions shown are not satisfied, the result may not hold. The main tool in our proofs is our comparison theorem, Theorem 3.1.

Theorem 5.1. *Let $A=M-N_1-N_2$ be a convergent regular composite splitting. Let $M=\hat{F}-\hat{G}=\tilde{F}-\tilde{G}$ be weak regular splittings and let $\hat{R}=\hat{F}^{-1}\hat{G}$, and $\tilde{R}=\tilde{F}^{-1}\tilde{G}$. Consider, as in Theorem 4.2, the iterative method defined by (17) with corresponding global iteration matrices $\hat{T}$ and $\tilde{T}$. If $\rho(\hat{R})\leqq\rho(\tilde{R})$ and $\hat{F}^{-1}\geqq\tilde{F}^{-1}$ then $\rho(\hat{T})\leqq\rho(\tilde{T})$.*

Proof. Let x be the Frobenius vector for a global iteration matrix T and let $\alpha=\rho(T)$, $\hat{\alpha}=\rho(\hat{T})$, and $\tilde{\alpha}=\rho(\tilde{T})$. Since $T=(I-F^{-1}N_2)^{-1}(R+F^{-1}N_1)\geqq R$, then, $\rho(R)\leqq\alpha$. Consider first the case $\hat{\alpha}=\rho(\hat{R})$. Then, $\hat{\alpha}=\rho(\hat{R})\leqq\rho(\tilde{R})\leqq\tilde{\alpha}$ and the theorem is proved. If, on the contrary, $\rho(\hat{R})<\hat{\alpha}$, or in our generic notation $\rho(R)<\alpha$, then, $(\alpha I-R)^{-1}$ exists and is nonnegative. Thus,

$$\begin{aligned}\alpha M_T x&=N_T x\\ \alpha(M(I-R)^{-1}-N_2)x&=M(I-R)^{-1}Rx+N_1x\\ \alpha(I-R)^{-1}x&=(I-R)^{-1}Rx+M^{-1}(N_1+\alpha N_2)x\\ (I-R)^{-1}(\alpha I-R)x&=M^{-1}(N_1+\alpha N_2)x\\ (I-R)^{-1}x&=(\alpha I-R)^{-1}M^{-1}(N_1+\alpha N_2)x.\end{aligned}$$

We may now replace the $(I-R)^{-1}x$ factor in $N_T x$ as follows:

$$\begin{aligned}N_T x&=M(I-R)^{-1}Rx+N_1x\\ &=MR(\alpha I-R)^{-1}M^{-1}(N_1+\alpha N_2)x+N_1x\\ &=GF^{-1}(\alpha I-GF^{-1})^{-1}(N_1+\alpha N_2)x+N_1x\geqq 0.\end{aligned}$$

Finally, since $\hat{F}^{-1}\geqq\tilde{F}^{-1}$, $\hat{M}_T^{-1}=(I-\hat{F}^{-1}N_2)^{-1}\hat{F}^{-1}\geqq(I-\tilde{F}^{-1}N_2)^{-1}\tilde{F}^{-1}=\tilde{M}_T^{-1}$, and thus, by Theorem 3.1, $\hat{\alpha}\leqq\tilde{\alpha}$. □

The following corollary shows that $\rho(T_p)$ is a monotonically decreasing function of p. The corollary applies, in particular, to two-stage methods, and to iterative block Gauss-Seidel.

Corollary 5.2. *Consider the solution of $Ax=b$ by a nested iteration (Definition 4.6). Let at each level $A=M-N_1-N_2$ be a convergent regular composite splitting and at the innermost level let $M=B-C$ be a weak regular splitting. Let q and p be nonnegative integers. Consider two nested iterations, differing only in the number of inner iterations at the outer iteration, p in one case and q in the other. If $q \geqq p$, then $\rho(T_q) \leqq \rho(T_p)$.*

Proof. If there are only two levels of nesting, i.e., the iteration is a two-stage method, with inner splitting $M=B-C$, then define $R=B^{-1}C$. Otherwise, define R as the iteration matrix for the inner nested iteration. From Corollary 4.7 it follows that R is convergent. Clearly $\rho(R^q) \leqq \rho(R^p)$. Consider the splittings induced on M by $R^p = F_p^{-1} G_p$ and $R^q = F_q^{-1} G_q$; see Lemma 2.3. Then,

$$\begin{aligned} F_q^{-1} &= (M(I-R^q)^{-1})^{-1} = (I-R^q)(I-R)^{-1}F^{-1} \\ &= \sum_{i=0}^{q-1} R^i F^{-1} \geqq \sum_{i=0}^{p-1} R^i F^{-1} = (M(I-R^p)^{-1})^{-1} = F_p^{-1}. \end{aligned}$$

Therefore by Theorem 5.1 $\rho(T_p) \geqq \rho(T_q)$. $\square$

As we pointed out, this corollary applies in particular to two-stage methods. Intuition indicates that if more inner iterations are performed, i.e., if we have a better approximation to be exact solution of (2) at each outer step, then the method should converge faster. A closer look at Example 4.4 reveals that

$$T_2 = \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.25 \end{bmatrix} \qquad T_3 = \begin{bmatrix} -0.125 & 0.75 & -0.125 \\ 0.75 & -0.25 & 0.75 \\ -0.125 & 0.75 & -0.125 \end{bmatrix}$$

and $\rho(T_2)=0.85$, $\rho(T_3)=1.3$. Thus if the conditions of Theorems 4.2 and 5.1 are violated, not only might there not be convergence for a small number of inner iterations, but the spectral radii of the iteration matrices may not be monotonically decreasing.

In the final result of this section, which follows directly from Lemma 3.2, we extend the point Stein-Rosenberg theorem [26] to the methods studied here. The name of the theorem is inspired by the following observation. If $\hat{N}_2=0$, we have an iterative (block) Jacobi type method, while the case $N_2 \neq 0$ yields an iterative block Gauss-Seidel type method.

Theorem 5.3 (Nested Stein-Rosenberg). *Let $A=M-N_1-N_2=M-\hat{N}_1-\hat{N}_2$ be convergent regular composite splittings. Consider the solution of $Ax=b$ by a nested iteration. Assume that the inner systems $Mx=r(x_k, x_{k+1}, b)$ are solved by a nested iteration with iteration matrix R and induced splitting $M=F-G$. Let the corresponding iteration matrices be $T=(F-N_2)^{-1}(G+N_1)$ and $\hat{T}=(F-\hat{N}_2)^{-1}(G+\hat{N}_1)$. If $N_2 \geqq \hat{N}_2$ then $\rho(T) \leqq \rho(\hat{T})$.*

6 Dynamic nested iteration

In this section we study Algorithm 4.1 in the case where $p_{ik}=p_k$ for all i. We call this algorithm Dynamic Nested Iteration. The main result of the section is Theorem 6.4, where we show that under certain conditions, this method is convergent. Dynamic nested iteration can be viewed as the concatenation of different nested methods, each with a different iteration matrix T_{p_k}. Thus, for r outer iterations, the global iteration matrix for the dynamic nested iteration is the product $T_{p_r} T_{p_{r-1}} \dots T_{p_1}$. The difficulty in analyzing this method is that the product of convergent matrices may not be convergent. The following definition and lemmas provide the background material for Theorem 6.4 and strengthen a result in [23].

It is well known, that for any matrix, A, there exists a permutation matrix P such that $L=PAP^T$ is block lower triangular, i.e. $L=(L_{ij})$, $i, j=1, \dots, q$, where L_{ii} is an $s_i \times s_i$ block either irreducible or a 1×1 null matrix and $L_{ij}=0$ for $j>i$. This form is called the reduced normal form of A and it is unique, up to permutations within the diagonal blocks and of the ordering of certain diagonal blocks, see, e.g., [3, 10, 13], or [26].

Definition 6.1. Let P be a permutation matrix for A, such that PAP^T is in reduced normal form. We say that B is Triangular Conformable with A under P, if $L_B=PBP^T$ is such that $(L_B)_{ij}=0$ for $j>i$.

Lemma 6.2. *Let P be a permutation matrix for the nonnegative matrix A, such that $L=PAP^T$ is in reduced normal form and let $\rho_b=\rho(L_{bb})$, $b=1, 2, \dots, q$. If for exactly one block, say a, $\rho(A)=\rho_a=\max_b \rho_b$, then the Frobenius vector for L has the form $w=(0, 0, \dots, 0, v_a, w_{a+1}, w_{a+2}, \dots, w_q)^T$, where v_a is the Frobenius vector for L_{aa}.*

Proof. Clearly $(Lw)_b=(\rho_a w)_b$ for all $b\leqq a$. For $b>a$ assume inductively that $(Lw)_c=(\rho_a w)_c$ for all $c<b$. Since $\rho_a>\rho_b$, it follows that $(\rho_a I-L_{bb})^{-1}$ exists and is nonnegative. Therefore $w_b=(\rho_a I-L_{bb})^{-1}\sum_{i=1}^{b-1} L_{bi} w_i$ is such that $\sum_{i=1}^{b} L_{bi} w_i =\rho_a w_b$. □

Lemma 6.3. *Let $T\geqq 0$, and let $\{T_i\}$ be a collection of convergent nonnegtive matrices such that T_i is Triangular Conformable to T under P for all i, with $L=PTP^T$ and $L^i=PT_iP^T$. Let v_a be the s_a-dimensional Frobenius vector of L_{aa}, with $\rho_a =\rho(L_{aa})<1$, for every a. If for every i, $L^i_{aa} v_a \leqq L_{aa} v_a=\rho_a v_a$ then, for any fixed set $\{i_1, \dots, i_r\}$, $\rho(\Upsilon_r)=\rho(T_{i_1} T_{i_2} \dots T_{i_r})\leqq \max_a \rho^r_a<1$.*

Proof. Since $\rho(\Upsilon_r)=\rho(P\Upsilon_r P^T)=\rho\left(\prod_{j=1}^r PT_{i_j} P^T\right)=\max_a \rho\left(\prod_{j=1}^r L^{i_j}_{aa}\right)$, there are two cases to be considered. If $L_{aa}=0$ then $\prod_{j=1}^r L^{i_j}_{aa}=0$. If $L_{aa}\neq 0$ then L_{aa} is irreducible and therefore $v_a>0$. It follows that $\prod_{j=1}^r L^{i_j}_{aa} v_a\leqq \rho^r_a v_a$ and therefore, by Lemma 2.1(d), $\rho\left(\prod_{j=1}^r L^{i_j}_{aa}\right)\leqq \rho^r_a<1$ for every a. Thus $\rho(\Upsilon_r)\leqq \max_a \rho^r_a<1$. □

Theorem 6.4. *If $A=M-N_1-N_2$ is a convergent regular composite splitting and $M=F-G$ is a regular splitting, then the Dynamic Nested Iterative Method with r iterations is convergent for every r. Moreover, the method is no slower than the case when $p_k=1$, $k=1,\dots,r$. In other words, $\rho(\Upsilon_r)\leqq\rho_1^r<1$, where $\Upsilon_r=T_{p_r}T_{p_{r-1}}\dots T_{p_1}$, T_{p_k} is the iteration matrix (13) corresponding to p_k, and $\rho_1=\rho(T_1)$.*

Proof. We have proved in Corollary 5.2 that $\rho(T_1)\geqq\rho(T_p)$ for all p. Let P be a permutation matrix such that PT_1P^T is in reduced normal form. The proof proceeds in two parts. First we show that T_p is triangular conformable to T_1 under P for all p. Then we show that if ρ_a, v_a is such that $\rho_a v_a=L_{aa}v_a$, then for every i, $L_{aa}^i v_a\leqq\rho_a v_a$ and by Lemma 6.3 $\rho(\Upsilon_r)=\rho(T_{p_r}T_{p_{r-1}}\dots T_{p_1})\leqq\max\limits_a\rho_a^r=\rho_1^r<1$.

Part 1: First consider T_1. Let $R=F^{-1}G$,

$$T_1=(I-F^{-1}N_2)^{-1}(F^{-1}N_1+R)\tag{18}$$
$$=F^{-1}N_1+R+\sum_{i=1}^{\infty}(F^{-1}N_2)^i(F^{-1}N_1+R).$$

Therefore, since $F^{-1}N_1$, R, and $F^{-1}N_2$ are nonnegative, no cancellation can occur. It follows then that

$$R,F^{-1}N_1,(F^{-1}N_2)^iF^{-1}N_1,(F^{-1}N_2)^iR\tag{19}$$

are triangular conformable to T_1 under P. Now consider T_p. Let

$$Q_p=(I-R^p)M^{-1}=(I-R^p)(I-R)^{-1}F^{-1}=\sum_{i=0}^{p-1}R^iF^{-1},\tag{20}$$

then

$$T_p=(I-Q_pN_2)^{-1}(Q_pN_1+R^p)$$
$$=\left(I-\sum_{i=0}^{p-1}R^iF^{-1}N_2\right)^{-1}\left(\sum_{i=0}^{p-1}R^iF^{-1}N_1+R^p\right)$$
$$=\left(\sum_{j=0}^{\infty}\left(\sum_{i=0}^{p-1}R^iF^{-1}N_2\right)^j\right)\left(\sum_{i=0}^{p-1}R^iF^{-1}N_1+R^p\right)$$

and since every instance of $(F^{-1}N_2)^j$ is followed by either $F^{-1}N_1$ or R^i $(i,j>0)$ it follows from (19) and the fact that triangular conformability is closed under multiplication and addition that T_p is triangular conformable to T_1 under P.

Part 2: Consider the splitting induced by T_p on $A=M(p)-N(p)$, with $M(p)=Q_p^{-1}-N_2$ and $N(p)=N_1+Q_p^{-1}R^p$; see Lemma 2.3. Then,

$$M(p)^{-1}=(I-Q_pN_2)^{-1}Q_p=\left(I-\sum_{i=0}^{p-1}R^iF^{-1}N_2\right)^{-1}\sum_{i=0}^{p-1}R^iF^{-1}\tag{21}$$
$$\geqq(I-F^{-1}N_2)^{-1}F^{-1}=M(1)^{-1}.$$

There are two cases to consider. If $L_{aa}=0$, for some a, then by Part 1, $L^p_{aa}=0$ for all p, and therefore $(P\Upsilon_r P^T)_{aa}=0$. We therefore assume that $L_{aa}\neq 0$ for all a. Let ρ_a, v_a such that $L_{aa}v_a=\rho_a v_a$. By Corollary 4.3 T_1 is convergent, thus $\rho_a<1$ for all a. Define q diagonal matrices $E(a)$, $a=1,2,\ldots,q$, called scaling matrices, where $E_b(a)$ is the $s_b\times s_b$ matrix $e(a,b)\,I_{s_b}$, $b=1,2,\ldots,q$, and $e(a,b)$ is defined by

$$\begin{aligned} e(a,b)&=1, \quad \text{if } \rho_b<\rho_a \quad \text{or } b=a\\ &=\rho_a+\frac{\dfrac{\rho_a}{\rho_b}-\rho_a}{2}, \quad \text{otherwise.}\end{aligned}$$

Consider $E(a)\,PT_1\,P^T$, $E(a)$ has the effect of scaling the b^{th} block row of $PT_1\,P^T$ by $e(a,b)\leqq 1$. Every diagonal block with a spectral radius less than ρ_a is left unchanged. Every diagonal block, b, with a spectral radius greater than ρ_a is scaled by $e(a,b)$ giving a new spectral radius $e(a,b)\,\rho_b=(\rho_b+1)\frac{\rho_a}{2}<\rho_a$. Thus $\rho(E(a)\,PT_1\,P^T)=\max_b\rho((E(a)\,PT_1\,P^T)_{bb})=\rho_a$. Let w_a be the Frobenius eigenvector of $E(a)\,PT_1\,P^T$, i.e., corresponding to ρ_a. Since all diagonal blocks except the a^{th} block have spectral radius less than ρ_a, then, by Lemma 6.2, $w_a^T=(0,0,\ldots,0,v_a,(w_a)_{a+1},\ldots,(w_a)_q)^T$. Since $M=F-G$ is a regular splitting, $N(1)=N_1+G\geqq 0$ and thus

$$\begin{aligned} E(a)\,PT_1\,P^T w_a&=\rho_a w_a\\ E(a)\,PM(1)^{-1}N(1)\,P^T w_a&=\rho_a w_a\\ N(1)\,P^T w_a&=\rho_a M(1)\,P^T E(a)^{-1} w_a\\ PM(1)\,P^T E(a)^{-1} w_a&=\frac{1}{\rho_a}PN(1)\,P^T w_a\geqq 0.\end{aligned}$$

We wish to show $PAP^T E(a)^{-1} w_a\geqq 0$.

$$\begin{aligned} PAP^T E(a)^{-1} w_a&=(PM(1)\,P^T E(a)^{-1}-PN(1)\,P^T E(a)^{-1})\,w_a\\ &=\left(\frac{1}{\rho_a}PN(1)\,P^T-PN(1)\,P^T E(a)^{-1}\right)w_a\\ &=PN(1)\,P^T\left(\frac{1}{\rho_a}I-E(a)^{-1}\right)w_a\geqq 0,\end{aligned}$$

where the last inequality follows from $1\geqq e(a,b)>\rho(a)$ for all a, b. From (21) it follows that $PM(p)^{-1}P^T\geqq PM(1)^{-1}P^T$, thus

$$\begin{aligned} PM(p)^{-1}P^T(PAP^T E(a)^{-1}w_a)&\geqq PM(1)^{-1}P^T(PAP^T E(a)^{-1}w_a)\\ PM(p)^{-1}AP^T E(a)^{-1}w_a&\geqq PM(1)^{-1}AP^T E(a)^{-1}w_a\\ P(I-T_p)\,P^T E(a)^{-1}w_a&\geqq P(I-T_1)\,P^T E(a)^{-1}w_a\\ PT_1\,P^T E(a)^{-1}w_a&\geqq PT_p\,P^T E(a)^{-1}w_a.\end{aligned}$$

Since $(PT_p\,P^T E(a)^{-1})_{aa}=L^p_{aa}$, and since $w_{a,j}=0$, for $j<a$, $(PT_p\,P^T E(a)^{-1}w_a)_a=L^p_{aa}v_a$ it follows that $\rho_a v_a=L_{aa}v_a\geqq L^p_{aa}v_a$ for all a. □

7 Nested block iterative methods

In this section we consider Algorithm 4.1. We first present conditions for convergence and for a monotonicity rule for the simpler case $p_{ik}=p_i$, for all k using the results from the previous sections. Later we show convergence for the general case.

Since F, G are block diagonal, $R_i=F_i^{-1}G_i$, where as before $R=F^{-1}G$. The equation to update a given component, i.e., the loop over j, may be represented as

$$x_{k+1}^{(i)}=(I_{s_i}-R_i^{p_{ik}})\,M_i^{-1}(b+N_1\,x_k+N_2\,x_{k+1})^{(i)}+R_i^{p_{ik}}\,x_k^{(i)}. \tag{22}$$

Let $\mathscr{P}_k=\{p_{1k}, p_{2k}, \ldots, p_{qk}\}$ and define

$$\mathscr{Q}(\mathscr{P}_k)=\begin{bmatrix} (I_{s_1}-R_1^{p_{i1}})\,M_1^{-1} & 0 & \ldots & 0 \\ 0 & (I_{s_2}-R_2^{p_{i2}})\,M_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (I_{s_q}-R_q^{p_{iq}})\,M_q^{-1} \end{bmatrix}, \tag{23}$$

$$\mathscr{R}(\mathscr{P}_k)=\begin{bmatrix} R_1^{p_{1k}} & 0 & \ldots & 0 \\ 0 & R_2^{p_{2k}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_q^{p_{qk}} \end{bmatrix}, \tag{24}$$

from where it follows that $\mathscr{Q}(\mathscr{P}_k)=(I-\mathscr{R}(\mathscr{P}_k))\,M^{-1}$. With these definitions it easy to see that the iteration matrix for the nested block iteration is given by:

$$H(\mathscr{P}_k):=(I-\mathscr{Q}(\mathscr{P}_k)\,N_2)^{-1}(\mathscr{Q}(\mathscr{P}_k)\,N_1+\mathscr{R}(\mathscr{P}_k)). \tag{25}$$

By Lemma 2.3,

$$M_H(\mathscr{P}_k)=\mathscr{Q}(\mathscr{P}_k)^{-1}-N_2, \qquad N_H(\mathscr{P}_k)=\mathscr{Q}(\mathscr{P}_k)^{-1}\,\mathscr{R}(\mathscr{P}_k)+N_1$$

define the unique splitting satisfying $A=M_H(\mathscr{P}_k)-N_H(\mathscr{P}_k)$, and $H(\mathscr{P}_k)=M_H(\mathscr{P}_k)^{-1}\,N_H(\mathscr{P}_k)$. In the following two theorems we assume $p_{ik}=p_i$, for all i, k.

Theorem 7.1. *If $M=F-G$ is a weak regular splitting, and $A=M-N_1-N_2$ is a convergent regular composite splitting, then $\rho(H(\mathscr{P}))<1$, where $H(\mathscr{P}_k)$ is defined in Eq.* (25).

Proof. The matrices $M_H(\mathscr{P})$ and $N_H(\mathscr{P})$ have the same form as M_T and N_T (Eqs. (14) and (15)), respectively, with $\mathscr{Q}(\mathscr{P})^{-1}$ replacing F and $\mathscr{Q}(\mathscr{P})^{-1}\,\mathscr{R}(\mathscr{P})$ replacing G. We will show that $M=\mathscr{Q}(\mathscr{P})^{-1}-\mathscr{Q}(\mathscr{P})^{-1}\,\mathscr{R}(\mathscr{P})$ is a weak regular splitting and the theorem will follow from Theorem 4.2. Clearly $\mathscr{Q}(\mathscr{P})$, and $\mathscr{R}(\mathscr{P})$ are nonnegative, so it remains to show that $\mathscr{Q}(\mathscr{P})^{-1}\,\mathscr{R}(\mathscr{P})\,\mathscr{Q}(\mathscr{P})\geqq 0$.

$$\begin{aligned} \mathscr{Q}(\mathscr{P})^{-1}\,\mathscr{R}(\mathscr{P})\,\mathscr{Q}(\mathscr{P}) &= M(I-\mathscr{R}(\mathscr{P}))^{-1}\,\mathscr{R}(\mathscr{P})(I-\mathscr{R}(\mathscr{P}))\,M^{-1} \\ &= M\mathscr{R}(\mathscr{P})\,M^{-1}=F\mathscr{R}(\mathscr{P})\,F^{-1} \\ &= \begin{bmatrix} (G_1\,F_1^{-1})^{p_1} & 0 & \ldots & 0 \\ 0 & (G_2\,F_2^{-1})^{p_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (G_q\,F_q^{-1})^{p_q} \end{bmatrix} \geqq 0. \quad \square \end{aligned}$$

Theorem 7.2. *Let $\mathscr{P}=\{p_1, p_2, \ldots, p_q\}$ and $\hat{\mathscr{P}}=\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_2\}$ where $p_i \leqq \hat{p}_i$ for all i. If $M=F-G$ is a weak regular splitting, and $A=M-N_1-N_2$ is a convergent regular composite splitting then $\rho(H(\hat{\mathscr{P}})) \leqq \rho(H(\mathscr{P}))$.*

Proof. Since $\mathscr{Q}(\hat{\mathscr{P}}) \geqq \mathscr{Q}(\mathscr{P})$ and $\rho(\mathscr{R}(\hat{\mathscr{P}})) \leqq \rho(\mathscr{R}(\mathscr{P}))$ the theorem follows from Theorem 5.1. □

We consider now the convergence of the nested block iterative method in the general case. The proof is analogous to that of Theorem 6.4.

Theorem 7.3. *Let $M=F-G$ be a regular splitting and $A=M-N_1-N_2$ be a convergent regular composite splitting. Algorithm 4.1 is convergent and no slower than the case when $p_{ik}=1$, $i=1, \ldots, q$, $k=1, \ldots, r$. In other words $\rho(\Upsilon_r) < \rho_1^r < 1$, for all r, where $\Upsilon_r = H(\mathscr{P}_r)\, H(\mathscr{P}_{r-1}) \ldots H(\mathscr{P}_1)$, and $\rho_1 = \rho(T_1)$.*

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