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Two Characterizations of Matrices with the Perron-Frobenius Property

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SUMMARY

Two characterizations of general matrices for which the spectral radius is an eigenvalue and the corresponding eigenvector is either positive or nonnegative are presented. One is a full characterization in terms of the sign of the entries of the spectral projector. In another case, different necessary and sufficient conditions are presented which relate to the classes of the matrix. These characterizations generalize well-known results for nonnegative matrices.

KEY WORDS: Perron-Frobenius property; generalization of nonnegative matrices

1. INTRODUCTION

We say that a real square matrix A has the *Perron-Frobenius property* if it has a *right Perron-Frobenius eigenvector*, i.e., if there exists a semipositive vector v for which $Av = \rho(A)v$, where $\rho(A)$ stands for the spectral radius of A. A semipositive vector is a nonzero nonnegative vector (see, e.g. [1]), where here and throughout the paper, the nonnegativity or positivity of vectors and matrices, and inequalities involving them, are understood to be componentwise. A left Perron-Frobenius eigenvector of A is the right Perron-Frobenius eigenvector of A^T, the transpose of A. Often the right (or left) Perron-Frobenius eigenvector is called a Perron eigenvector, or simply a Perron vector.

Matrices with the Perron-Frobenius property include positive matrices, and irreducible nonnegative matrices. In these cases the Perron eigenvector is positive and the spectral radius is a strictly dominant eigenvalue, i.e., the spectral radius is the only eigenvalue with the largest modulus. Nonnegative matrices, not necessarily irreducible, also have the Perron-Frobenius property. This follows from the Perron-Frobenius theorem [8], [18]. This theorem

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forms the basis for extensive work on nonnegative matrices and a variety of applications including the theory of matrix splitting; see, e.g., the monographs [1], [13], [19], [22]. Other matrices which have the Perron-Frobenius property are eventually positive (respectively, nonnilpotent eventually nonnegative) matrices, i.e., those matrices A whose powers A^k are positive (respectively, nonzero and nonnegative) for all integers $k \geq k_0$ for some positive integer k_0. These matrices were studied, e.g., in [3], [7], [10], [11], [14], [15], [23], [24].

There is also interest in general matrices (which may not be eventually nonnegative) having the Perron-Frobenius property; see, e.g., [6], [10], [12], [14], [16], [21]. Splittings for these matrices were studied in [5], [16], [17].

Two sets of interest are PFn and WPFn. The first is the set of $n \times n$ real matrices A such that both A and A^T have positive Perron-Frobenius eigenvectors and whose spectral radius is a simple positive and strictly dominant eigenvalue. The second, WPFn, is the set of $n \times n$ real matrices A such that both A and A^T have the Perron-Frobenius property. Several properties of these sets were studied in [6], including some topological properties.

In this paper, we present two new characterizations of these sets. In section 2, we present complete characterizations in terms of the sign of the entries of the spectral projectors. In section 3, we analyze the classes of matrices with the Perron-Frobenius property. We show different necessary and sufficient conditions for a matrix to be in PFn or WPFn.

2. Spectral decomposition and the Perron-Frobenius property

In this section, we give a characterization of all matrices in PFn, and of the matrices in WPFn with a simple and strictly dominant eigenvalue, in terms of the positivity or nonnegativity of their spectral projectors.

We were inspired to look for this characterization by the fact that for irreducible stochastic matrices (see, e.g., [1], [2]) it holds that the spectral projector is positive.

We first point out that the set PFn is equal to the set of $n \times n$ real matrices that are eventually positive [24], and that PFn is a proper subset of WPFn [6].

The following theorem is a known result. Its proof can be derived from the usual spectral decomposition that can be found, e.g., in [4, page 27] or [20, pages 114, 225] and its method of proof is similar to that of [24, Theorem 3.6]. Moreover, in this theorem and in the proof of Theorem 2.2 below, the index of a matrix A stands for the smallest nonnegative integer k such that rank $A^k =$ rank A^{k+1}, and $G_\lambda(A)$ denotes the generalized eigenspace of A corresponding to the eigenvalue λ.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ have d distinct eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d|$. Let P be the projection matrix onto $G_{\lambda_1}(A)$ along $\bigoplus_{j=2}^{d} G_{\lambda_j}(A)$ (P is called the spectral projector) and let $Q = A - \lambda_1 P$. Then, $PQ = QP$ and $\rho(Q) \leq \rho(A)$. Furthermore, if the index of $A - \lambda_1 I$ is 1 then $PQ = 0$.*

Theorem 2.2. *The following statements are equivalent:*

(i) $A \in PFn$.
(ii) $\rho(A)$ is an eigenvalue of A and in the spectral decomposition $A = \rho(A)P + Q$ we have $P > 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$.

Proof. Suppose that $A \in$ PFn. Then, A (respectively, A^T) has a positive or a negative eigenvector v (respectively, w) corresponding to the simple positive and strictly dominant eigenvalue $\rho = \rho(A)$. Suppose that v and w are normalized so that $v^T w = 1$. In the spectral decomposition, we have $A = \rho(A)P + Q$ where $P = vw^T$ and $\rho(Q) < \rho(A)$. Since $v^T w = 1$, it follows that the vectors v and w are either both positive or both negative. Therefore, $P = vw^T > 0$, which is obviously of rank 1. Conversely, suppose that $\rho = \rho(A)$ is an eigenvalue of A and that in the spectral decomposition $A = \rho(A)P + Q$, we have $P > 0$, rank $P = 1$ and $\rho(Q) < \rho(A)$. Since rank P =1, it follows that the algebraic multiplicity of ρ is 1. Thus, the index of $A - \rho I$ is 1 and by Theorem 2.1 we conclude that $PQ = QP = 0$. Therefore,

$$\left(\frac{1}{\rho}A\right)^k = \left(P + \frac{1}{\rho}Q\right)^k = P^k + \left(\frac{1}{\rho}Q\right)^k = P + \left(\frac{1}{\rho}Q\right)^k,$$

and consequently,

$$\lim_{k\to\infty}\left(\frac{1}{\rho}A\right)^k = P + \lim_{k\to\infty}\left(\frac{1}{\rho}Q\right)^k = P > 0.$$

Since $\rho > 0$ and the matrix $\frac{1}{\rho}A$ is real and eventually positive, it follows that the matrix A is also real and eventually positive. Thus, $A \in$ PFn. □

The proofs of the following two results are very similar to that of Theorem 2.2, and are therefore omitted.

Theorem 2.3. *The following statements are equivalent:*

(i) $A \in$ *WPFn has a simple and strictly dominant eigenvalue.*

(ii) $\rho(A)$ *is an eigenvalue of* A *and in the spectral decomposition* $A = \rho(A)P + Q$ *we have* $P \geq 0$, *rank* $P = 1$ *and* $\rho(Q) < \rho(A)$.

Theorem 2.4. *Let one of the two real matrices* A *and* A^T *possess the strong Perron-Frobenius property but not the other. Then, the projection matrix* P *in the spectral decomposition of* A *satisfies the relation* $P = vw^T$ *where one of the vectors* v *and* w *is positive and the other one is nonpositive.*

Example 2.5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then the Jordan canonical form of A is given by

$$J(A) = V^{-1}AV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$V = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad V^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Thus, $\rho(A) = 2$ is a simple positive and strictly dominant eigenvalue of A with a corresponding eigenvector $v = Ve_1 = [1\ 1\ 1]^T$, where $e_1 = [1\ 0\ 0]^T$ is the first canonical vector. Hence, A

possesses the strong Perron-Frobenius property. The matrix A^T has $\rho(A^T) = \rho(A) = 2$ as a simple positive and strictly dominant eigenvalue but with a corresponding eigenvector w where $w^T = e_1^T V^{-1} = [1\ 1\ -1]$. Thus, A^T does not possess the strong Perron-Frobenius property, and therefore, $A \notin$ PFn. In fact, the spectral projector is

$$P = vw^T = [1\ 1\ 1]^T[1\ 1\ -1] = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix},$$

where $v > 0$ and w is nonpositive, consistent with Theorem 2.4.

3. The classes of a matrix in WPFn

In this section, we first review some definitions, and then present some results that generalize a well-known result about the classes of a nonnegative matrix.

Let $\Gamma(A)$ denote the usual directed graph of a real matrix A; see, e.g., [1], [13]. A vertex j has access to a vertex m if there is a path from j to m in $\Gamma(A)$. If j has access to m and m has access to j, then we say j and m communicate. If n is the order of the matrix A, then the communication relation is an equivalence relation on $\langle n \rangle := \{1, 2, \ldots, n\}$ and an equivalence class is called a *class* of the matrix A[†]. If in the graph $\Gamma(A)$, a vertex m has access to a vertex j which happens to be in a class α of matrix A, then we say m has access to α. If $A \in \mathbb{R}^{n\times n}$ and α, β are ordered subsets of $\langle n \rangle$, then $A[\alpha, \beta]$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by β according to the prescribed orderings of the ordered sets α and β. For simplicity, we write $A[\alpha]$ for $A[\alpha, \alpha]$. If A is in $\mathbb{R}^{n\times n}$ and $\kappa = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is an ordered partition of $\langle n \rangle$ (see, e.g., [2], [3]), then A_κ denotes the block matrix whose $(i, j)^{th}$ block is $A[\alpha_i, \alpha_j]$. A class of A is called an *initial class* if it is not accessed by any other class of A, and it is called a *final class* if it does not access any other class of A. If α is a class of A for which $\rho(A[\alpha]) = \rho(A)$, then we call α a *basic class*. We note here that the concept of a basic class is normally used for a square nonnegative matrix [1], and that it is justifiable to extend this definition to an arbitrary square real matrix because, in view of the Frobenius normal form of A (see, e.g., [2], [9]), for every class α, we have $\rho(A[\alpha]) \le \rho(A)$. (In other words, the possibility that $\rho(A[\alpha]) > \rho(A)$ is ruled out.)

We are ready to discuss the following well-known result; see, e.g., [1, Ch. 2, Theorem (3.10)].

Theorem 3.1. *A nonnegative matrix A has a positive right eigenvector corresponding to $\rho(A)$ if and only if the final classes of A are exactly its basic ones.*

We first show that a matrix in PFn must be irreducible, and hence, it has one class which is basic, final, and initial. Then, we study analogous necessary and sufficient conditions on the classes of a matrix so that it is in WPFn. Unlike the nonnegative case, these necessary and sufficient conditions are not the same. They are presented in two complementary theorems.

Theorem 3.2. *If A is a matrix in PFn, then A is irreducible. Hence, A has one class, which is basic, final, and initial.*

[†]A class is thus the same as a strong component of the directed graph of the matrix. In the literature on nonnegative matrices, the name class is usually preferred; see, e..g, [1].

Proof. Let $\kappa = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be an ordered partition of the set of vertices $\langle n \rangle$ that gives the lower triangular Frobenius normal form of A. It is enough to show that $m = 1$. Assume to the contrary that $m > 1$, then $A[\alpha_1, \alpha_m] = 0$ which implies that $((A_\kappa)^s)[\alpha_1, \alpha_m] = 0$ for all positive integers s. Thus, A_κ (and consequently A) can not be eventually positive, a contradiction. Hence, A has only one class, which is basic, final, and initial. □

Theorem 3.3. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that $\rho(A)$ is an eigenvalue (which holds in particular if A has the Perron-Frobenius property). If α is a final class of A and $v[\alpha]$ is nonzero for some right eigenvector v of A corresponding to $\rho(A)$, then α is a basic class.*

Proof. In general, for any class α of A, we have

$$(Av)[\alpha] = A[\alpha]v[\alpha] + \sum_{\beta} A[\alpha, \beta]v[\beta],$$

where the sum on the right side is taken over all classes β that are accessed from α but are different from α. When α is a final class and v is an eigenvector of A corresponding to $\rho(A)$, we have

$$\rho(A)v[\alpha] = (Av)[\alpha] = A[\alpha]v[\alpha].$$

If, in addition, $v[\alpha]$ is nonzero, we can conclude that α is a basic class. □

The proofs of Theorem 3.4 and Corollary 3.5 are similar to Theorem 3.3, and therefore, are omitted.

Theorem 3.4. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix such that $\rho(A)$ is an eigenvalue (which holds in particular if A^T has the Perron-Frobenius property). If α is an initial class of A and $w[\alpha]$ is nonzero for some left eigenvector w of A corresponding to $\rho(A)$, then α is a basic class.*

Corollary 3.5. *Let A be a square matrix such that $\rho(A)$ is an eigenvalue (which holds in particular if $A \in$ WPFn).*

(i) If α is a final class of A and $v[\alpha]$ is nonzero for some right eigenvector v of A corresponding to $\rho(A)$, then α is a basic class.
(ii) If α is an initial class of A and $w[\alpha]$ is nonzero for some left eigenvector w of A corresponding to $\rho(A)$, then α is a basic class.

Theorem 3.6. *If $A \in \mathbb{R}^{n \times n}$ has a basic and initial class α for which $A[\alpha]$ has a right Perron-Frobenius eigenvector, then A has the Perron-Frobenius property.*

Proof. There is a semipositive vector v such that $A[\alpha]v = \rho(A)v$. Define the vector $\tilde{v} \in \mathbb{R}^n$ as follows: for any class γ of A, $\tilde{v}[\gamma] = v$ if $\gamma = \alpha$, and $\tilde{v}[\gamma] = 0$ if $\gamma \neq \alpha$. It is easily seen that $\tilde{v}$ is semipositive and that $A\tilde{v} = \rho(A)\tilde{v}$. □

The proofs of Theorem 3.7 and Corollary 3.8 are similar to Theorem 3.6, and thus, are omitted.

Theorem 3.7. *If $A \in \mathbb{R}^{n \times n}$ has a basic and final class β for which $A[\beta]$ has a left Perron-Frobenius eigenvector, then A^T has the Perron-Frobenius property.*

Corollary 3.8. *If $A \in \mathbb{R}^{n \times n}$ has two classes α and β, not necessarily distinct, such that:*

α is basic, initial, and $A[\alpha]$ has a right Perron-Frobenius eigenvector
β is basic, final, and $A[\beta]$ has a left Perron-Frobenius eigenvector,

then $A \in$ *WPFn.*

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