Problem Set 3

(Out Mon 02/20/2023, Due Thu 03/16/2023)

Problem 5

Consider the linear ODE system

$$\begin{cases} \vec{u}'(t) = A \cdot \vec{u}(t) \\ \vec{u}(0) = \mathring{\vec{u}} \end{cases} \tag{1}$$

where

$$A = \begin{pmatrix} -5000 & 4999 & 0 & 0 \\ 4999 & -5000 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 0 & 0 & 10 & 0 \end{pmatrix} \quad \text{and} \quad \mathring{\vec{u}} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

We would like to approximate the solution of (1) on $t \in [0, 1]$, using an ODE solver with equidistant time steps. We are happy to be within 5% accuracy.

(a) Calculate the true solution $\vec{u}(t)$.

(b) Implement the following time-stepping schemes and apply them to this problem:

1. forward Euler
2. backward Euler
3. RK4
4. Crank-Nicolson
5. Adams-Bashforth 4
6. Adams-Moulton 4
7. BDF2
8. BDF4

[For multistep methods, simply cheat and use the correct solution for the first $r-1$ steps.]
Then determine, for each scheme, numerically the maximum time step that yields the desired accuracy.

(c) Provide a discussion of your observations, explaining for each scheme the reason(s) for the resulting time step size needed. Moreover, explain any interesting patters, such as a high order scheme doing worse than a low order scheme.

(d) For the largest time step that yields the desired accuracy for backward Euler, $k_{\text{BE}}^{\max}$ plot the numerical solutions obtained with each scheme, when using that same time step $k_{\text{BE}}^{\max}$ (plot also the true solution in the same figure, and limit the u-axis to $[-2, 2]$). Explain your observations.

Problem 6

Consider the r-step BDF method of the form

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k f(U^{n+r}) . \tag{2}$$

(a) Write a short Matlab program that for each r automatically computes the vector of coefficients $\vec{\alpha} = (\alpha_0, \dots, \alpha_r)$, such that (2) is globally r-th order accurate.

(b) Plot the regions of absolute stability for the BDF methods $r \in \{1, \dots, 8\}$.

(c) What do those plots reveal about zero-stability?

(d) Describe how the regions of absolute stability behave with increasing order (e.g., do they grow or shrink, what parts of the left half plane do they contain).

(e) Apply all 8 schemes (BDF1 to BDF8, using exact starting values for the first $r-1$ steps) to the test problem

$$\begin{cases} \vec{u}'(t) = A \cdot \vec{u}(t) \\ \vec{u}(0) = \mathring{\vec{u}} \end{cases}$$

where

$$A = \begin{pmatrix} 161 & 581 & -999 & -580 \\ -181 & -401 & 999 & 780 \\ 19 & -181 & -1 & -200 \\ -19 & 181 & -999 & -800 \end{pmatrix} \quad \text{and} \quad \mathring{\vec{u}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

on $t \in [0, 1]$, and plot the error convergence (error vs. step size in log–log) for 200 k-values within $10^{-3} \leq k \leq 10^{-1}$. Explain your observations, in particular explain the origin of the humps in BDF4, BDF5, and BDF6.

(f) Does an 8-step method (i.e., $r = 8$) exist of the form (2), but itself not being a BDF method, that is 7th order accurate and zero-stable? If yes, find one and plot its region of absolute stability.

Instructions

For each problem set, you need to submit one document, either in class or via email to the course instructor, that contains plots and explanations (hand-written or typed). If you decide to email the document, name it `yourfamilyname_problemset1.pdf`, where 1 stands for the number of the problem set.

In addition, for each programming task, email your respective program to the course instructor, under the filename `yourfamilyname_problem1a.m`, where `1` stands for the problem number and `a` for the sub-problem letter.