Problem Set 4

(Out Tue 11/02/2021, Due Tue 11/16/2021)

Problem 9

For the 2D heat equation

$$u_t = u_{xx} + u_{yy}$$

on the unit square $\Omega =]0,1[^2$, with homogeneous Dirichlet b.c. ($u = 0$ on $\partial\Omega$), consider the Alternating Direction Implicit (ADI) method

$$\begin{aligned} U^{n+\frac{1}{2}} &= U^n + \frac{k}{2}(D_x^2 U^{n+\frac{1}{2}} + D_y^2 U^n) \ , \\ U^{n+1} &= U^{n+\frac{1}{2}} + \frac{k}{2}(D_y^2 U^{n+1} + D_x^2 U^{n+\frac{1}{2}}) \ . \end{aligned}$$

(1) Use von Neumann stability analysis to investigate the stability properties of the ADI method. Is the method unconditionally stable, conditionally stable (if so, with which condition), or always unstable?

(2) Modify the program `temple8023_heateqn2d.m` on the course website to use the ADI method instead of the (default) Crank-Nicolson method. Then compare Crank-Nicolson with the ADI method for different numbers of time steps (`nt = 4,8,16,32`), in terms of errors and CPU times needed to conduct the computation. Plot both in log-log scale as functions of h.

Problem 10

Consider the advection-reaction equation

$$u_t + u_x = r(u) + s(x,u)$$

on $x \in [0, 2\pi[$ with periodic boundary conditions, and zero initial conditions $u(x,0) = 0$. The solution $u(x,t)$ represents a chemical concentration ($0 \le u \le 1$), which is advected with constant velocity, and modified by a bistable reaction term $r(u) = u(1-u)(u-\frac{1}{2})$ and a localized source term $s(x,u) = a \exp\left(-10(x-\pi)^2\right)(1-u)$, where $a > 0$ is a parameter.

(1) Write a program that approximates the true solution with sufficient accuracy, and run the simulation on the two cases $a = 0.5$ and $a = 1$. Plot both solutions at times $t \in \{2, 8, 40\}$. Explain your observations.

(2) There is a critical threshold value a_c, such that for $a < a_c$, the solution behaves like the case $a = 0.5$, and for $a > a_c$, the solution behaves like the case $a = 1$. Find a_c numerically, up to at least 0.1% accuracy. Remember that your scheme's global truncation error must be sufficiently small.