

Problem Set 2

(Out Thu 09/09/2021, Due Thu 09/23/2021)

Problem 4

Consider the 1d Poisson equation

$$\begin{cases} -u_{xx} = f & \text{in }]0, 1[\\ u = 0 & \text{on } \{0, 1\} \end{cases} \quad (1)$$

with $f(x) = \sin(\phi(x))(\phi_x(x))^2 - \cos(\phi(x))\phi_{xx}(x)$, where $\phi(x) = 9\pi x^2$.

Run the program `mit18336_poisson1d_error.m` given on the course web site, which approximates (1) by a sequence of linear systems, based on the approximation $u_{xx} \approx D^2u$, where $D^2u(x) = \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h))$.

- (1) Explain the observed error convergence rate.
- (2) Modify the system matrix, such that fourth order error convergence is achieved. Show error convergence plots.
- (3) Return to the original system matrix based on $u_{xx} \approx D^2u$. Now change the right hand side vector from $f_i = f(ih)$ to $f_i = f(ih) + \frac{h^2}{12}D^2f(ih)$. Prove that this modification yields fourth order accuracy, and produce an error convergence plot that verifies this result.¹ How does the error constant compare to fourth order system matrix in part (2)?
- (4) Change the right hand side to

$$f(x) = \begin{cases} 1 & \text{for } x \leq \frac{1}{2} \\ 0 & \text{for } x > \frac{1}{2} \end{cases}$$

and report and explain the new error convergence rate for the three solution approaches.

Problem 5

Write a program that approximates the solution of the *biharmonic equation*

$$\begin{cases} u_{xxxx} = f & \text{in }]0, 1[\\ u = 0 & \text{on } \{0, 1\} \\ u_x = 0 & \text{on } \{0, 1\} \end{cases}$$

with $f = 24$. Perform a numerical error analysis for your approximation, and report and explain your observations.

¹This trick is called *deferred correction*.