Temple 8023 **Numerical Differential Equations I** **Fall 2019**

Problem Set 2

(Out Wed 09/18/2019, Due Wed 10/02/2019)

Problem 4

Consider the 1d Poisson equation

$$\begin{cases} -u_{xx} = f & \text{in }]0,1[\\ u = 0 & \text{on } \{0,1\} \end{cases} \tag{1}$$

with $f(x) = \sin(\phi(x))(\phi_x(x))^2 - \cos(\phi(x))\phi_{xx}(x)$, where $\phi(x) = 9\pi x^2$.
Run the program `mit18336_poisson1d_error.m` given on the course web site, which approximates (1) by a sequence of linear systems, based on the approximation $u_{xx} \approx D^2u$,
where $D^2u(x) = \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h))$.

(1) Explain the observed error convergence rate.

(2) Modify the system matrix, such that fourth order error convergence is achieved. Show error convergence plots.

(3) Return to the original system matrix based on $u_{xx} \approx D^2u$. Now change the right hand side vector from $f_i = f(ih)$ to $f_i = f(ih) + \frac{h^2}{12}D^2f(ih)$. Prove that this modification yields fourth order accuracy, and produce an error convergence plot that verifies this result.[1] How does the error constant compare to fourth order system matrix in part (2)?

(4) Change the right hand side to

$$f(x) = \begin{cases} 1 & \text{for } x \le \frac{1}{2} \\ 0 & \text{for } x > \frac{1}{2} \end{cases}$$

and report and explain the new error convergence rate for the three solution approaches.

Problem 5

Write a program that approximates the *biharmonic equation*

$$\begin{cases} u_{xxxx} = f & \text{in }]0,1[\\ u = 0 & \text{on } \{0,1\} \\ u_x = 0 & \text{on } \{0,1\} \end{cases}$$

with $f = 24$. Perform a numerical error analysis for your approximation, and report and explain your observations.

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[1] This trick is called *deferred correction.*