Temple 8024 Numerical Differential Equations II Fall 2010

Problem Set 3

(Out Thu 10/14/2010, Due Thu 10/28/2010)

Problem 6

---

(1) Download the code `temple8024_weno_claw.m` from the course website and run it. Plot the numerical approximation obtained with 80 cells, together with the true solution.

(2) Change the code so that it approximates the linear advection equation with smooth initial conditions. Perform a numerical error analysis and report the scheme's convergence rate. Explain why we do not obtain fifth order, even though a fifth order WENO reconstruction is used.

Problem 7

Write an at least third order accurate WENO code[1] that solves the 2D advection equation

$$\phi_t + u\phi_x + v\phi_y = 0\ , \quad (x,y) \in]0,1[^2, t \in]0,T[$$

with the velocity field $(u,v) = (-\psi_y, \psi_x)$ where $\psi(x) = \cos(\pi t/T)\sin^2(\pi x)\sin^2(\pi y)$, with $T = 4$.

Run your code on the initial conditions $\phi(x,y,0) = \sqrt{(x-0.25)^2 + (y-0.3)^2} - 0.1$, and plot the zero contour $\Gamma(t) = \{x : \phi(x,t) = 0\}$ at times $t \in \{0,1,2,3,4\}$. Do so for three resolutions: one for which the results look bad, one for which they look descent, and one for which they look very good.

Problem 8

Consider the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \text{ on } x \in [-1,1[\tag{1}$$

with periodic boundary conditions. Here, the dispersive term u_{xxx} prevents waves from breaking. The KdV equation is a nonlinear equation that possesses smooth traveling wave solutions, so called *solitons*

$$u(x,t) = f_c(x - ct)\ , \tag{2}$$

where $f_c(x) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}x\right)$. Here $\text{sech}(x) = \frac{2}{e^x + e^{-x}}$. It is a nice exercise in differentiation to verify that for every $c > 0$, (2) solves (1).

Write a code that approximates (1). I recommend an explicit (conservative) treatment of the nonlinear advection term, and an implicit treatment of the dispersion term. Test your code on the following sets of initial data

(a) $u_0(x) = f_{400}(x)$

(b) $u_0(x) = f_{400}(x+0.7) + f_{200}(x)$

(c) $u_0(x) = \frac{1}{2}\left(f_{400}(x+0.7) + f_{200}(x)\right)$

Plot the results at time $t = 0.015$. Explain your observations in all the three cases. In particular explain how the nonlinear nature of (1) becomes visible in cases (b) and (c). Then run your code for case (b) up to $t = 0.1$, and see if you can get reasonably close to the true solution.

---

[1] Use the code `temple8024_weno_claw.m` for inspiration.