Mori-Zwanzig formalism

- Conditional expectation \( \mathcal{E} \) and orthogonal projection \( F = I - \mathcal{E} \).
- Solution operator \( e^{tRF} \) and orthogonal dynamics solution operator \( e^{tRF} \) satisfy Duhameil’s principle (Dyson’s formula).

\[
e^{tRF} = \int_0^t e^{(t-s)RF}REdfds + e^{tRF}.
\]

- Differentiating Dyson’s formula:

\[
i\partial_t e^{tRF} = RE\partial_t e^{tRF} + \int_0^t e^{(t-s)RF} RREdfds + e^{tRF}.\
\]

- Adding \( E \) from right yields evolution for average solution operator

\[
i\partial_t e^{tRF} = \mathcal{E} e^{tRF} + \int_0^t K(t-s)E e^{tRF} ds,
\]

where \( \mathcal{E} = RE \) and \( K(t) = e^{tRF} \) memory kernel.

Evolution for average solution:

\[
i\partial_t (\mathcal{E} e^{tRF}) = \mathcal{E} e^{tRF} + \int_0^t K(t-s)\mathcal{E} e^{tRF} ds,
\]

where \( \mathcal{E} = RE \) and \( K(t) = e^{tRF} \) memory kernel.

\[
\mathcal{E} e^{tRF} = \int_0^t e^{(t-s)RF}REdfds + e^{tRF}.
\]

- \( \mathcal{E} = RE \) and \( K(t) = e^{tRF} \) memory kernel.

Approximations for radiative transfer

Here consider uncorrelated measure, i.e. covariance matrix \( A \) diagonal.

\[
\mathcal{E} e^{tRF} = \int_0^t e^{(t-s)RF}REdfds + e^{tRF}.
\]

- Omitting the memory term: \( i\partial_t \hat{u}(t) = \hat{R}(t) \) yields classical \( P_N \) closure.

- Piecewise constant quadrature for memory:

\[
\int_0^t K(t-s)\mathcal{E} e^{tRF} ds \approx K(t)\mathcal{E} e^{tRF}.
\]

With characteristic time scale \( \tau = \frac{\lambda}{\pi^2} \) yields classical diffusion closure correction [4]

\[
i\partial_t \hat{u}(t) = \hat{R}(t) + \frac{1}{\tau} \nabla \cdot \nabla \hat{u}(t).
\]

Better approximation for short times:

\[
\hat{R}(t) = \hat{R}(t) + \min(1, t) \nabla \cdot \nabla \hat{u}(t).
\]

Yields new crescendo diffusion correction closure (no extra cost).

(Explicit time dependence models loss of information.)