COVERS OF SURFACES, KLEINIAN GROUPS, AND THE CURVE COMPLEX

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ABSTRACT. We prove an effective version of a theorem relating curve complex distance to electric distance in hyperbolic 3-manifolds, up to errors that are polynomial in the complexity of the underlying surface. We use this to give an effective proof of a result regarding maps between curve complexes of surfaces induced by finite covers. As applications, we effectively relate the electric circumference of a fibered manifold to the curve complex translation length of its monodromy, and we give quantitative bounds on virtual specialness for cube complexes dual to curves on surfaces.

1. Introduction

Let $S$ be an orientable surface of finite type with negative Euler characteristic. The curve graph $\mathcal{C}(S)$ of $S$ is the graph whose vertices are homotopy classes of essential simple closed curves and whose edges correspond to pairs of such homotopy classes that admit disjoint representatives. A finite-sheeted cover $p: \tilde{S} \to S$ induces a (coarsely well-defined) map $p^*: \mathcal{C}(S) \to \mathcal{C}(\tilde{S})$ sending a vertex $\gamma$ of $\mathcal{C}(S)$ to its full preimage $p^{-1}(\gamma) \subset \tilde{S}$. In [RS09], Rafi–Schleimer show that the map $p^*$ is a $C$-quasi-isometric embedding, with $C$ depending only on $\deg(p)$, the degree of $p$, and on $\chi(S)$. Their result roughly implies that “pairs of simple closed curves do not detangle very much under pull-back by finite covers of small degree,” leading us to pose the following question:

**Question 1.** Given simple closed curves $\alpha$ and $\beta$ on $S$, what is the minimal degree of a cover $\tilde{S}$ of $S$ such that there exist disjoint components $\tilde{\alpha}, \tilde{\beta}$ of the pre-images of $\alpha, \beta$, respectively?

Unfortunately, this question cannot be answered using [RS09] because their techniques do not pin down how the constant $C$ depends on $\deg(p)$ and $\chi(S)$. Therefore, our approach to the question is to prove the following theorem:

**Theorem (7.1).** Let $p: \tilde{S} \to S$ be a finite covering map between non-sporadic surfaces $\tilde{S}, S$. Then for any $\alpha, \beta$ distinct essential simple closed curves on $S$,

$$\frac{d_{\mathcal{C}(S)}(\alpha, \beta)}{\deg(p) \cdot A_3(|\chi(S)|)} \leq d_{\mathcal{C}(\tilde{S})}(p^*(\alpha), p^*(\beta)) \leq d_{\mathcal{C}(S)}(\alpha, \beta),$$

where $A_3$ is the polynomial $A_3(x) = 80 e^{54} \pi x^{13}$ when $S$ is closed.

When $S$ has punctures,
\[
\frac{d_{C(S)}(\alpha, \beta)}{\deg(p)^4 \cdot A_3(|\chi(S)|)} \leq d_{C(\tilde{S})}(p^*(\alpha), p^*(\beta)) \leq d_{C(S)}(\alpha, \beta),
\]
where \(A_3\) is the polynomial \(A_3(x) = 416e^{114\pi^{37}}x^{37}\).

Moreover, the linear dependence on \(\deg(p)\) in the first bound of Theorem 7.1 is sharp—see Remark 7.2 for more details.

The polynomial \(A_3\) is a product \(A_1 \cdot A_2\) of polynomials

\[
A_1(x) = \begin{cases} 
20e^{44}x^{10} & \text{for } S \text{ closed,} \\
104e^{94}\pi^{30}x^{30} & \text{otherwise}
\end{cases}
\]

(1)

and

\[
A_2(x) = \begin{cases} 
4e^{10}\pi x^{3} & \text{for } S \text{ closed,} \\
4e^{20}\pi^{7}x^{7} & \text{otherwise,}
\end{cases}
\]

which arise independently of one another (e.g., see Theorem 4.1 below) and we will refer to them often in what follows.

The main ingredient in proving Theorem 7.1 is the following theorem regarding the relationship between curve graph distance and electric distance in hyperbolic 3-manifolds. Throughout the paper, we use the same notation for a simple closed curve, its corresponding vertex of the curve graph, and its geodesic representative in a 3-manifold whenever it is clear through context which of the three we are referring to. In Section 3 we define a constant \(\epsilon_S\) which is bounded from below by a polynomial of degree 2 in \(\frac{1}{|\chi(S)|}\) when \(S\) is closed and degree 6 in \(\frac{1}{|\chi(S)|}\) in general. This is used in the following theorem.

**Theorem (4.1).** Let \(\alpha\) and \(\beta\) be curves in \(S\) and let \(M\) be a complete hyperbolic structure on \(S \times \mathbb{R}\) such that \(\ell_M(\alpha), \ell_M(\beta) \leq \epsilon_S\). Then

\[
\frac{1}{A_1(|\chi(S)|)} \cdot d_{C(S)}(\alpha, \beta) \leq d^*_M(\alpha, \beta) \leq A_2(|\chi(S)|) \cdot d_{C(S)}(\alpha, \beta),
\]

where the polynomials \(A_1\) and \(A_2\) are as in Equation (1), and \(d^*_M\) is the metric obtained from the hyperbolic metric \(d_M\) by electrifying the \(\epsilon_S\)-thin part of \(M\).

In [Tan12], Tang used the original, non-effective (i.e. where the dependence on \(|\chi(S)|\) was not explicit) version of Theorem 4.1 to reprove the Rafi-Schleimer result, and we follow his argument to obtain Theorem 7.1 from Theorem 4.1.

The non-effective version of Theorem 4.1 is originally due to Bowditch [Bow11, Theorem 5.4]. (See also the statement of Theorem 4.1 in Biringer–Souto [BS15].) As Biringer–Souto state, in reference to Theorem 4.1, “credit should also be given to Yair Minsky, since [the theorem] is implicit in the development of the model manifolds of [Min10], and to Brock–Bromberg [BB11], who prove a closely related result.” However, it is important to note that all of the proofs of these results rely on compactness arguments, which cannot be made effective in a way that is necessary for our applications. Thus, the main contribution of Theorem 4.1 is that it gives the explicit
relationship between curve complex and electric distance. Instead of relying on compactness arguments using geometric limits, we argue using \(1\)-Lipschitz sweepouts in \(M\) (see Theorem 2.1 in Section 2).

Indeed, although there are by now many results relating geometric invariants of hyperbolic manifolds to combinatorial invariants of curves on surfaces, almost none of these can quantify or estimate the exact dependence on the complexity of the underlying surface. Moreover, even when such dependence has been estimated, it is usually (at least) exponential in \(|\chi(S)|\). For example, Brock’s theorems relating volumes of hyperbolic manifolds to distances in the pants graph [Bro03a, Bro03b] are prime examples of important results relating geometry to combinatorics where dependence on the surface was left completely undetermined. However, in forthcoming work of the first and third author with Webb [ATW18], this dependence is bounded using a careful analysis of Masur–Minsky hierarchies [MM00], but the bound produced is on the order of \(|\chi(S)|^{13}\). Hence, one major novelty of Theorem 4.1 is that our error terms are explicit and depend polynomially on \(|\chi(S)|\). To our knowledge, the only other such results are due to Futer–Schleimer [FS14] who estimate the cusp area of a hyperbolic manifold in terms of translation length in the arc complex.

Using Theorem 7.1 we address Question 1 by giving a lower bound on \(\deg_{p} \alpha, \beta \), the minimal degree of a cover necessary to have disjoint components \(\tilde{\alpha}, \tilde{\beta}\) of the preimages of \(\alpha, \beta\), respectively. We emphasize again that this application requires the effective statement of Theorem 7.1 proven here.

**Corollary 1.1.** For two simple closed curves \(\alpha\) and \(\beta\) on a surface \(S\),

\[
\frac{d_{C(S)}(\alpha, \beta)}{C(|\chi(S)|)} \leq \deg(\alpha, \beta)^a,
\]

where \(C(x) = 3A_1(x)A_2(x)\) is a polynomial of degree 13 and \(a = 1\) when \(S\) is closed. In general, \(C(x)\) is a polynomial of degree 37 and \(a = 4\).

In Section 8, we provide an application of this corollary to certain virtually special cube complexes. Given a sufficiently complicated collection \(\Gamma\) of curves on a closed surface \(S\), Sageev’s construction [Sag95] gives rise to a dual CAT(0) cube complex on which \(\pi_1(S)\) acts freely and properly discontinuously. The quotient of the complex under this action is a non-positively curved cube complex \(\mathcal{C}_\Gamma\). It is well known by the work of Haglund–Wise [HW08] that \(\mathcal{C}_\Gamma\) is virtually special, meaning that it has a finite degree cover whose fundamental group embeds nicely into a right-angled Artin group. The following theorem quantifies this statement by estimating the degree of the required cover in the case that \(\Gamma\) is a pair of curves:

**Theorem (8.3).** Suppose that \(\alpha\) and \(\beta\) are two simple closed curves that together fill a closed surface \(S\). Let \(\deg_{\mathcal{C}_\Gamma}\) be the minimal degree of a special cover of the dual cube complex \(\mathcal{C}_\Gamma\) to the curve system \(\Gamma = \alpha \cup \beta\). Then

\[
\frac{d_{C(S)}(\alpha, \beta)}{C(|\chi(S)|)} \leq \deg_{\mathcal{C}_\Gamma},
\]

where \(C(x) = 3A_1(x)A_2(x)\) is a polynomial of degree 13.
Theorem 8.3 is related to work of M. Chu [Chu17] and J. Deblois, N. Miller, and the second author [DMP18] on quantifying virtual specialness for various hyperbolic manifolds.

As a second application, we use Theorem 4.1 to effectively relate the electric circumference of a fibered manifold $M_\phi$ to the curve graph translation length $\ell_S(\phi)$ of its monodromy $\phi: S \to S$. (For definitions, see Section 9.)

**Theorem (9.1).** If $\phi: S \to S$ is a pseudo-Anosov homeomorphism of a closed surface $S$, then

$$\frac{1}{A_1(\chi(S))} \cdot \ell_S(\phi) \leq \text{circ}_e(M_\phi) \leq A_2(\chi(S)) \cdot (\ell_S(\phi) + 2),$$

where the polynomials $A_1$ and $A_2$ are as in Equation (1).

The outline of the paper is as follows. In Section 2 we give the necessary background on curve graphs, Margulis tubes in hyperbolic manifolds, pleated surfaces and sweepouts. We then prove various lemmas regarding curves on surfaces and Margulis tubes in 3–manifolds in Section 3 before proving Theorem 4.1 in Sections 4, 5, and 6 and Theorem 7.1 in Section 7. In Section 8 we prove the application regarding cube complexes and in Section 9 we prove the application to fibered manifolds.

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2. **Background**

Given an orientable surface $S$ of genus $g \geq 0$, $n \geq 0$ punctures, and without boundary, define $\omega(S) := 3g + n - 4$. We call an orientable, boundary-less surface of finite type $S$ non-sporadic if $\omega(S) > 0$. In all that follows, in order to avoid trivial cases we will assume that all surfaces are non-sporadic.

2.1. **Curves on surfaces.** Recall that a simple curve is essential if it is neither null-homotopic nor peripheral (i.e. it doesn’t bound a disk or once punctured disk on $S$). As is usual in the subject, we will generally refer to a vertex $\alpha \in C(S)^{(0)}$ as a curve. We reserve the term loop to refer to an embedded circle in $S$. For example, with these conventions, a curve is represented by a loop.

Given curves $\alpha, \beta \in C(S)^{(0)}$, their geometric intersection number, denoted $i(\alpha, \beta)$, is defined as

$$i(\alpha, \beta) = \min \{|a \cap b| \},$$
where the minimum is taken over loop representatives \(a\) and \(b\) of \(\alpha\) and \(\beta\), respectively. A surgery argument due to Hempel [Hem01] yields the following upper bound on distance in \(\mathcal{C}(S)\) in terms of geometric intersection number:

\[
d_{\mathcal{C}(S)}(\alpha, \beta) \leq 2 \log_2(i(\alpha, \beta)) + 2.
\]

In particular, this shows that \(\mathcal{C}(S)\) is connected.

More recently, Bowditch [Bow14] proved a stronger version of Equation (3) which is sensitive to the topology of the underlying surface, and which we will need in subsequent sections. We reformulate Corollary 2.2 of [Bow14] as follows:

\[
d_{\mathcal{C}(S)}(\alpha, \beta) < 2 + 2 \cdot \frac{\log(i(\alpha, \beta)/2)}{\log((|\chi(S)| - 2)/2)},
\]

so long as the denominator is well-defined and positive which is the case for all \(S\) with \(|\chi(S)| \geq 5\).

### 2.2. Hyperbolic manifolds and Margulis tubes.

Here we review some basic background on hyperbolic 2- and 3-manifolds. For additional details, see [Bus10].

Let \(S\) be a finite area hyperbolic surface and let \(p\) be a puncture of \(S\). A peripheral curve about \(p\) corresponds to a parabolic element of \(\pi_1(S)\). A horodisk in \(H^2\) based at a lift \(\tilde{p}\) of the puncture \(p\) will project to a neighborhood of \(p\) in \(S\). There exists a horocycle \(\tilde{Q}_p\) such that the quotient of \(\tilde{Q}_p\) by \(\text{stab}(\tilde{p}) < \pi_1(S)\) is not embedded, and that for any horocycle \(\tilde{H}\) based at \(\tilde{p}\), \(\tilde{H}/\text{stab}(\tilde{p})\) is embedded. We call \(\tilde{Q}_p\) the maximal horocycle for \(\tilde{p}\). The region of \(S\) facing \(p\) and bounded by the quotient \(Q_p\) of \(\tilde{Q}_p\) is called the maximal cusp neighborhood.

There is another horocycle \(\tilde{H}_p\) based at \(\tilde{p}\) which projects to an embedded loop of length 2. The region bounded between this loop and \(p\) is called a standard cusp neighborhood. The standard cusp neighborhood is isometric to the cylinder \((-\infty, \log 2) \times S^1\) equipped with the metric

\[
dx^2 + e^{2x}d\theta^2,
\]

where \(-\infty \leq x \leq \log(2)\) and \(\theta \in S^1 = [0, 1]/(0 \sim 1)\) (see pages 110-112 of [Bus10]).

A key feature of hyperbolic geometry is that the volume of an \(n\)-dimensional ball of radius \(r\), denoted \(\text{Vol}_n(r)\), grows exponentially as a function of \(r\). In subsequent sections we will need explicit formulae for this volume in low dimensions, so we record that information here:

\[
\text{Vol}_2(r) = 4\pi \sinh^2(r/2), \quad \text{Vol}_3(r) = \pi(\sinh(2r) - 2r).
\]

In particular,

\[
\text{Vol}_2(r) = O(e^r), \quad \text{Vol}_3(r) = O(e^{2r}),
\]

and the following limits exist:

\[
\lim_{r \to 0} \frac{\text{Vol}_2(r)}{r^2}, \quad \frac{\text{Vol}_3(r)}{r^3}.
\]

Given a hyperbolic manifold \(M\) and \(\delta > 0\), the \(\delta\)-thin part of \(M\), denoted by \(M_{(0, \delta]}\), is the set of points in \(M\) with injectivity radius at most \(\delta/2\). Similarly, the \(\delta\)-thick...
part, \(M_{[\delta, \infty]}\), consists of all points with injectivity radius at least \(\delta/2\). Any hyperbolic manifold \(M\) is of course the union of its \(\delta\)-thick and \(-\)-thin parts.

For \(n \geq 2\), there exists \(\epsilon_n > 0\) called the \(n\)-dimensional Margulis constant so that the \(\epsilon_n\)-thin part of any hyperbolic \(n\)-manifold \(M\) decomposes into a disjoint union of cusps and subsets of the form \(T_{\alpha_1}, T_{\alpha_2}, \ldots, T_{\alpha_n}\) where \(T_{\alpha_i}\) is a tubular neighborhood of the closed geodesic \(\alpha_i\).

Thus the \(\epsilon_2\)-thin part of a hyperbolic surface is homeomorphic to a disjoint union of annuli, and the \(\epsilon_3\)-thin part of a hyperbolic 3-manifold decomposes as a disjoint union of solid tori and cusps.

Meyerhoff [Mey87] demonstrated the following lower bound for \(\epsilon_3\), which we will subsequently need:

\[
\epsilon_3 > 0.104.
\]

Given \(\delta \leq \epsilon_3\), we denote by \(T_{\alpha}(\delta)\) the component of \(M_{(0, \delta]}\) containing the geodesic \(\alpha\). This is called the Margulis tube for \(\alpha\). We note that \(T_{\alpha}(\delta)\) can be empty if the length of \(\alpha\) is greater than \(\delta\). We note here that there is a definite distance between \(T_{\alpha}(\epsilon_3)\) and \(\partial T_{\alpha}(\epsilon_3)\), which goes to infinity as \(\delta \to 0\). A concrete estimate of this growth was obtained recently by Futer–Purcell–Schleimer [FPS18]. We will require this estimate in Section 3 and so we record it there in detail.

We now briefly discuss hyperbolic 3-manifolds \(M = \mathbb{H}^3/\Gamma\) homeomorphic to \(S \times \mathbb{R}\), as these manifolds are the focus of this paper. Here and throughout, we always consider such a manifold with a fixed homotopy equivalence \(\iota: S \to M\), called a marking, which allows us to identify homotopy classes of curves in \(S\) with homotopy classes in \(M\). Hence, for any essential curve \(\alpha\) in \(S\) it makes sense to speak of its length \(\ell_M(\alpha)\) in \(M\), which is defined to be the length of the geodesic in \(M\) homotopic to \(\iota(\alpha) \subset M\).

Associated to any such manifold \(M\) without accidental parabolics\(^1\) is a pair of end invariants \((\lambda^-, \lambda^+)\), each of which is either:

1. (non-degenerate) a point in the Teichmüller space of \(S\), namely a pair \((f, \sigma)\) where \(\sigma\) is a complete hyperbolic metric on \(S\) and \(f: S \to \sigma\) is a (homotopy class of) homeomorphism;
2. (degenerate) a lamination on \(S\).

End invariants describe the behavior of the geometry of \(M = \mathbb{H}^3/\Gamma\) as one exits out of each of the two topological ends \(\mathcal{E}^-, \mathcal{E}^+\) of \(M\). In the non-degenerate case, an end \(\mathcal{E}\) of \(M\) is foliated by surfaces \(S_t\) equipped with induced metrics that converge (after a re-scaling) to a hyperbolic metric on \(S\).

In the degenerate case, Thurston [Thu78] proved that there exists a sequence of simple closed curves on \(S\) whose geodesic representatives exit \(\mathcal{E}\), and which converge, in the proper sense, to a lamination on \(S\). That an end, in general, behaves in exactly one of the above two ways follows from work of Bonahon [Bon86] and Canary [Can93], and ultimately the proof of the Tameness conjecture by Agol [Ago04] and Calegari–Gabai [CG06].

\(^1\)That is, where parabolics of \(\Gamma\) all come from peripheral loops in \(\pi_1(S)\).
The celebrated Ending Lamination Theorem, proved by Minsky [Min10] and Brock-Canary-Minsky [BCM12], asserts that the end invariants \((\lambda^+, \lambda^-)\) associated to \(E^+, E^-\), respectively, determines the hyperbolic manifold \(M\).

### 2.3. Pleated surfaces and sweepouts.

Fixing a hyperbolic 3-manifold \(M\), a topological surface \(S\), and a lamination \(\lambda\) on \(S\), a \(\lambda\)-pleated surface is a map \(F : S \to M\) so that:

1. \(F\) is proper, and hence sends cusps to cusps,
2. for each leaf \(l\) of \(\lambda\), \(F(l) \subset M\) is a geodesic,
3. for each component \(R\) of \(S \setminus \lambda\), \(F(R)\) is totally geodesic.

The map \(F\) induces via pull-back a complete hyperbolic metric on the surface \(S\). With respect to this metric, \(F\) is a 1-Lipschitz map of hyperbolic manifolds. Pleated surfaces arise very naturally in the study of hyperbolic 3-manifolds. For example, the convex core of a quasi-Fuchsian hyperbolic 3-manifold is always bounded by the image of a pair of pleated surfaces. Moreover, if \(M\) is homeomorphic to \(S \times \mathbb{R}\), work of Thurston [Thu78] implies that if all leaves of \(\lambda\) can be realized as geodesics in \(M\), there exists a \(\lambda\)-pleated surface into \(M\).

Given a surface \(S\) and a hyperbolic 3-manifold \(M\), a 1-Lipschitz sweepout is a homotopy \(f_t : X_t = (S, g_t) \to M\), where \(g_t\) is a hyperbolic metric on \(S\) and \(f_t\) is a 1-Lipschitz map for each \(t\). Whenever \(M\) is homeomorphic to \(S \times \mathbb{R}\), we only consider 1-Lipschitz maps homotopic to our fixed marking. An important result of Canary [Can96] yields the existence of 1-Lipschitz sweepouts interpolating between geodesics in \(M\).

**Theorem 2.1** (Canary). Let \(M = \mathbb{H}^3 / \Gamma\) be a hyperbolic 3-manifold homeomorphic to \(S \times \mathbb{R}\), so that \(\rho \in \Gamma\) is parabolic if and only if it corresponds to a peripheral curve on \(S\). Let \(\alpha, \beta\) be a pair of simple closed curves on \(S\) with geodesic representatives \(\alpha^*, \beta^*\) in \(M\). Then there exists a 1-Lipschitz sweepout \(f_t : X_t \to M, 0 \leq t \leq 1\), so that \(\alpha^* \subset f_0(X_0)\) and \(\beta^* \subset f_1(X_1)\).

Moreover, there exists a 1-Lipschitz sweepout \(g_t : S_t \to M, -\infty < t < \infty\), surjecting onto \(M\) and so that \(g_t(X_t)\) exits out of \(E^+\) (resp. \(E^-\)) as \(t \to \infty\) (resp. \(-\infty\)).

Theorem 2.1 follows from Canary’s work on *simplicial hyperbolic surfaces* [Can96]. These are path-isometric mappings into \(M\) of singular hyperbolic surfaces with cone points coinciding with vertices of a geodesic triangulation. Concretely, Canary shows the existence of 1-Lipschitz sweepouts where \(g_t\) is a simplicial hyperbolic surface. Brock [Bro03a] reformulates Canary’s construction by uniformizing and replacing each \(g_t\) with the unique non-singular hyperbolic metric in its conformal class. For additional details, see the proof of Lemma 4.2 in [Bro03a].

### 3. Hyperbolic surfaces and 3-manifolds

In this section we cover some fairly basic results in hyperbolic geometry. While nothing here will be surprising to experts, care must be taken in order to keep track of how the constants involved depend on the underlying parameters.

Let \(\hat{S}_0\) denote the compact subset of \(S\) obtained by deleting neighborhoods of each cusp consisting of points with injectivity radius at most \(\delta/2\).
Lemma 3.1. Let $S$ be an orientable surface with $\chi(S) < 0$ which is not a 3-times punctured sphere. Then for any $\delta \geq 0$ there is a constant $L_{S,\delta}$ such that for any finite area hyperbolic structure on $S$ and any $x$ in $\hat{S}_\delta$, there is an essential loop in $S$ through $x$ of length less than $L_{S,\delta}$.

Proof. Let $\tilde{x}$ be any lift of $x$ to the universal cover $\tilde{S} = \mathbb{H}^2$, and let $\pi : \mathbb{H}^2 \to S$ denote the universal covering map. Let $\tilde{B}$ be a lift of a maximal embedded open ball centered at $x$. By maximality, there must be a pair of points $z, y$ on the boundary of $\tilde{B}$ which lie in the same fiber over $S$. Then if $[a, b]$ represents the geodesic segment with endpoints $a, b \in \mathbb{H}^2$,

$$\tilde{\rho} := \pi([\tilde{x}, z]) \ast \pi([y, \tilde{x}])$$

is a loop $\rho$ representing a non-trivial element of $\pi_1(S, p)$. If $\rho$ is not simple, there will be a representative for an essential simple loop through $x$ which is contained in the image of $\rho$ (and whose length is therefore bounded above by that of $\rho$), and we replace $\rho$ with this.

Recall that the area of a hyperbolic disk of radius $r > 0$ is $4\pi \sinh^2(r/2)$, and therefore by the Gauss-Bonnet theorem the radius of $\tilde{B}$ is at most

$$2 \sinh^{-1}\left(\sqrt{|\chi(S)|}/2\right).$$

Hence the length of $\tilde{\rho}$ is at most

(10) \hspace{1cm} \ell_S := 4 \sinh^{-1}\left(\sqrt{|\chi(S)|}/2\right) = 4 \log\left(\sqrt{|\chi(S)|}/2 + \sqrt{1 + |\chi(S)|}/2\right).$$

When $S$ is closed, this concludes the proof. When $S$ has cusps, the above argument gives us the desired loop unless the element of $\pi_1(S, x)$ represented by $\rho$ is peripheral about a puncture, $p$. In this case, recall that $Q_p$ (resp. $H_p$) denotes the quotient of a maximal horocycle $\tilde{Q}_p$ (resp. a standard horocycle $\tilde{H}_p$).

Suppose first that $x$ lies in the standard cusp neighborhood. Since $x \in \hat{S}_\delta$, (5) implies that the distance $d_S(x, H_p)$ between $x$ and $H_p$ satisfies

$$e^{-d_S(x, H_p)} \geq \delta,$$

and therefore

(11) \hspace{1cm} d_S(x, H_p) \leq \log(1/\delta).$$

Let $N_p$ be the subset of the maximal cusp neighborhood bounded by $Q_p$ and $H_p$. Since the area of the neighborhood of a cusp is equal to the length of its boundary, by the Gauss-Bonnet theorem we have that $Q_p$ has length at most $2\pi|\chi(S)|$. The region $N_p$ can be lifted to a rectangle $\tilde{N}_p$ in the upper half-plane model which is (up to isometry) of the form

$$\tilde{N}_p = \{(y, z) \in \mathbb{H}^2 : 0 \leq y < a, 0 < r \leq z \leq b\},$$

for some positive $a$ and $b$. Then $H_p$ lifts to the top edge of $\tilde{N}_p$ and $Q_p$ lifts to the bottom edge. Therefore,

$$\ell(H_p) = \frac{a}{b} = 2; \ell(Q_p) = \frac{a}{r} \leq 2\pi|\chi(S)|.$$
Hence $a = 2b$ and so $r \geq b/\pi \chi(S)$, and so
\begin{equation}
(12) \quad d_S(H_p, Q_p) \leq \log(|\chi(S)|).
\end{equation}

By maximality, a fundamental lift $\tilde{Q}_p$ of $Q_p$ will project to the concatenation of (at least) two loops $\rho_1$ and $\rho_2$ on $S$. It cannot be the case that both $\rho_i$ are peripheral since otherwise $S$ would be a thrice punctured sphere. Moreover, there is a representative of $\rho_i$ of length at most $2 \log(|\chi(S)|) + 2$; indeed, if $[\tilde{y}_i, \tilde{z}_i]$ is a geodesic segment projecting to $\rho_i$, by (12) there is a path from $\tilde{y}_i$ to $\tilde{H}_p$ of length at most $\log(|\chi(S)|)$. Concatenating this with a segment of $\tilde{\rho}$ as necessary and then (applying (12) again) with a segment back to $\tilde{z}_i$ yields the desired bound. Abusing notation slightly, we refer to these representatives as $\rho_1, \rho_2$ and we note also that $\rho_i$ is contained completely within $\tilde{N}_p$ and $\rho_i$ touches $H_p$.

Since $x$ lies within the standard cusp neighborhood, $x$ must be within a distance of at most $2 + \log(1/\delta)$ from each $\rho_i$, and therefore there is an essential loop through $x$ (which, using the same argument as above we can assume is simple) of length at most
\[ 2(2 + \log(1/\delta)) + 2 \log(|\chi(S)|) + 2. \]
If $x \in N_p$, it can be at most $\log(|\chi(S)|) + 2$ from each $\rho_i$ and thus there is an essential loop through $x$ of length at most
\[ 2(\log(|\chi(S)|) + 2) + 2 \log(|\chi(S)|) + 2 = 4 \log(|\chi(S)|) + 6. \]

It remains to consider the case that $x$ is separated from the puncture by $Q_p$. Recall the simple loop $\rho$ constructed in the first part of the argument, and that we are assuming that $\rho$ is peripheral. We claim that $\rho$ must touch $Q_p$. Indeed, let $\tilde{p} \in \partial_x(\mathbb{H}^2)$ denote a lift of the puncture $p$ and let $\tilde{Q}_p$ be the horocycle based at $\tilde{p}$ projecting to $Q_p$. Since $\rho$ is peripheral, there is a lift $\tilde{\rho}$ bounded by lifts $\tilde{x}_1, \tilde{x}_2$ of $x$ so that $\tilde{x}_1$ and $\tilde{x}_2$ are on the same horocycle $R$ based at $\tilde{p}$. By maximality of $\tilde{Q}_p$, there is another lift $\tilde{Q}_p'$ of $Q_p$ that is tangent to $\tilde{Q}_p$ and which intersects $R$ at two points (see Figure 1).

Letting $g \in \pi_1(S)$ be the parabolic element corresponding to the peripheral loop $\rho$, all translates of $\tilde{x}_1$ under the action of $g$ on $\mathbb{H}^2$ lie along $R$ and lie outside of all lifts of the horoball $\tilde{Q}_p$ bounded by $Q_p$ since $x$ is separated from $p$ by $Q_p$. Therefore, there exists a lift $\tilde{\rho}'$ of $\rho$ with endpoints $\tilde{x}_3$ and $\tilde{x}_4$ that lie along $R$ such that $\tilde{Q}_p \cup \tilde{Q}_p'$ separate $\tilde{x}_3$ and $\tilde{x}_4$ (again see Figure 1). Thus, $\tilde{\rho}'$ must intersect $\tilde{Q}_p \cup \tilde{Q}_p'$ so that $\rho$ touches $Q_p$.

Since the length of $\rho$ is at most $\ell_S$, it follows that $x$ must be a distance of at most $\ell_S/2$ from $Q_p$, and hence from the region $N_p$. Thus, there is an essential simple loop through $x$ of length at most $\ell_S + 4 \log(|\chi(S)|) + 6$, and so in all three cases the loop constructed has length at most
\[ L_{S, \delta} = 2 \log(1/\delta) + \ell_S + 4 \log(|\chi(S)|) + 6. \]

\[ \Box \]

**Remark 3.2.** We use the proof of Lemma 3.1 to give an upper bound on $L_{S, \delta}$ that will be useful in subsequent sections. First we use (10) to give the following upper bound
Figure 1. When \( \rho \) is separated from \( p \) by \( Q_p \), it must touch \( Q_p \).

on \( \ell_S \):

\[
\ell_S \leq 4 \log(2\sqrt{|\chi(S)|}) = 4 \log(2) + 2 \log(|\chi(S)|) \leq 4 + 2 \log(|\chi(S)|).
\]

Note the proof of Lemma 3.1 is much simpler when \( S \) is closed and we can use \( \ell_S \) in place of \( L_{S,\delta} \) in this case. Combining (13) and the definition of \( L_{S,\delta} \) in the general setting we have:

\[
L_{S,\delta} \leq 2 \log(1/\delta) + 6 \log(\pi |\chi(S)|) + 9.
\]

Additionally, when \( S \) has punctures, we let \( L_S \) denote \( L_{S,\epsilon_3} \), which by (9) is at most

\[
6 \log(\pi |\chi(S)|) + 14.
\]

When \( S \) is closed we set \( L_S = \ell_S \), which by (13) is at most

\[
2 \log(|\chi(S)|) + 4.
\]

**Lemma 3.3.** Let \( M \) be a hyperbolic manifold with \( M \cong S \times \mathbb{R} \), and let \( \alpha \) be an essential curve on \( S \). There is a positive constant \( \epsilon_S < \epsilon_3 \) such that if \( f: S \to M \) is a \( \pi_1 \)-injective 1-Lipschitz map such that \( f(S) \cap \mathbb{T}_\alpha(\epsilon_S) \neq \emptyset \), then \( \ell_S(\alpha) \leq L_S \).

Further, there is a loop in the isotopy class of \( \alpha \) whose length is less than \( L_S \) in \( S \) and whose image in \( M \) is contained in \( \mathbb{T}_\alpha(\epsilon_3) \).

**Proof.** Given a positive \( \mu < \epsilon_3 \) and a non-empty \( \mu \)-tube \( \mathbb{T}_\alpha(\mu) \), let \( \mathcal{F}_\alpha(\mu) \) denote the distance between the boundary of the Margulis tube and \( \mathbb{T}_\alpha(\mu) \). The function \( \mathcal{F}_\alpha \) is decreasing in \( \mu \), and Theorem 1.1 of Futer–Purcell–Schleimer [FPS18] states that

\[
\mathcal{F}_\alpha(\mu) \geq \mathcal{F}(\mu) := \text{arccosh} \frac{\epsilon_3}{\sqrt{7.256 \mu}} - .0424.
\]

Let \( \epsilon_S = \mathcal{F}^{-1}(L_S/2) \). Hence (15) implies

\[
\mathcal{F}(\epsilon_S) = L_S/2 = \text{arccosh} \frac{\epsilon_3}{\sqrt{7.256 \epsilon_S}} - .0424,
\]

and thus,

\[
\epsilon_S = \frac{\epsilon_3^2}{7.256 \cdot \cosh^2(L_S/2 + .0424)}.
\]
Applying the upper bounds on $L_S$ obtained at the end of Remark 3.2 and (9), we see that
\begin{equation}
\epsilon_S \geq \frac{\epsilon_3^2}{8 \cdot \cosh^2 (\log(|\chi(S)|) + 2)} \geq \frac{1}{e^{10} \cdot |\chi(S)|^2} \tag{16}
\end{equation}
when $S$ is closed, and
\begin{equation}
\epsilon_S \geq \frac{\epsilon_3^2}{8 \cdot \cosh^2 (3 \log(\pi |\chi(S)|) + 7)} \geq \frac{1}{e^{20} \pi^6 \cdot |\chi(S)|^6} \tag{17}
\end{equation}
when $S$ has punctures. The lower bound on $\epsilon_S$ obtained above does not play a direct role in the proof of Lemma 3.3, but we will refer to this bound in later sections.

If $f(x) \in \mathbb{T}_\alpha(\mu)$ for $\mu < \epsilon_3$, then $x \in S_{\epsilon_3}$ since the $\epsilon_3$-thin part of any cusp neighborhood must map via $f$ into the $\epsilon_3$-thin part of a cusp neighborhood of $M$, and any Margulis tube in $M$ is disjoint from all $\epsilon_3$-thin cusp neighborhoods in $M$. Thus by Lemma 3.1 there is an essential simple loop $\rho$ through $x$ of length at most $L_S$.

Since the map is 1–Lipschitz, $f(\rho)$ has length less than $L_S$ and meets $\mathbb{T}_\alpha(\epsilon_S)$. Hence, any point on $f(\rho)$ has distance at most $L_S/2$ from $\mathbb{T}_\alpha(\epsilon_S)$, and so by construction $f(\rho) \subset \mathbb{T}_\alpha(\epsilon_3)$.

As $f$ induces an isomorphism on $\pi_1$, $f(\rho)$ is homotopic to some power of $\alpha$. But, $\rho$ is a simple curve on $S$ and so we must have that $\rho$ and $\alpha$ are isotopic curves on $S$. Set $\alpha = \rho$ and note that $\ell_S(\alpha) = \ell_S(\rho) \leq L_S$. \hfill $\square$

**Remark 3.4.** Notice in the proof that $\epsilon_S$ is chosen so that it is bigger than the quantities in Equations (16) and (17), but also small enough in comparison to $\epsilon_3$ so that the distance between the $\epsilon_S$-tube and the boundary of the $\epsilon_3$-tube is at least $F(\epsilon_S) \geq 2$.

**Lemma 3.5.** There is a universal constant $D$ so that if $\gamma_1$ and $\gamma_2$ are essential loops on $S$ with length less than $L_S$, then $d_{C(S)}(\gamma_1, \gamma_2) \leq D$.

**Proof.** By the collar lemma, $\gamma_1$ has an embedded annular neighborhood of width at least
\begin{equation}
\log(\coth(\ell(\gamma_1))/4) \coloneqq c(\gamma_1).
\end{equation}
Since $\log(\coth(x/4)) \to \infty$ as $x \to 0$ and decays to 0 as $x \to \infty$, there is some positive constant $c$ so that
\begin{equation}
c/2 = \log(\coth(c/4)).
\end{equation}
By inspection we see that $c < 2$.

We first present a proof in the special case that $S$ is closed, as in this setting the argument is more conceptual:

**$S$ is closed:** Assume that $\gamma_1$ is the shortest curve on $S$, and let $\mathcal{N}(\gamma_1)$ denote a maximal collar neighborhood of $\gamma_1$. Let $\bar{\mathcal{N}}$ denote a lift of the boundary of $\mathcal{N}(\gamma_1)$ to the universal cover. By maximality, there is a pair of points $\bar{x}, \bar{y} \in \bar{\mathcal{N}}$ which project to the same point on the boundary of $\mathcal{N}(\gamma_1)$. If $\bar{x}, \bar{y}$ lie on opposite boundary components, the geodesic segment connecting them must project to an essential loop $\rho$, since it intersects $\gamma_1$ exactly once. Moreover, in this case the width of $\mathcal{N}(\gamma_1)$ must be at least $\ell(\gamma_1)/2$, for
if not there will be a representative of $\rho$ with length less than $\ell(\gamma_1)$. Indeed, $\rho$ is the loop which lifts to $\tilde{\rho}$ which begins at $\tilde{x}$ with the perpendicular between $\tilde{x}$ and $\tilde{\gamma}_1$, then travels at most half of $\tilde{\gamma}_1$ (wrapping around if necessary), and then concludes with the perpendicular to $\tilde{\gamma}_1$ containing $\tilde{y}$.

If $\tilde{x}, \tilde{y}$ lie on the same boundary component $E$ of $\tilde{N}$, a loop $\rho$ is constructed in the exact same fashion as at the end of the previous paragraph, although it is now no longer the case that $\rho$ intersects $\gamma_1$ once. However $\rho$ must still be essential, for if not, $\rho$ would lift to the boundary of a geodesic triangle with three non-ideal vertices and two right angles. Since $S$ is closed, $\rho$ is non-peripheral and $\gamma_1$ must admit a collar neighborhood of width at least $\ell p \gamma_1 q{2}$ as in the previous paragraph.

Thus, $\gamma_1$ must admit a collar neighborhood of width at least $\ell p \gamma_1 q{2}$. Therefore, $i(\gamma_2, \gamma_1) \leq \min[2\ell(\gamma_2)/\ell(\gamma_1), \ell(\gamma_2)/c(\gamma_1)]$.

Thus, if $\ell(\gamma_1) > 2$, $i(\gamma_1, \gamma_2) \leq L_S$, and if $\ell(\gamma_1) < 2$, $c(\gamma_1) > 1/2$, so we have $i(\gamma_1, \gamma_2) \leq 2L_S$.

Consider the case where $|\chi(S)| \geq 5$. Using (4), we note that if $d_{C(S)}(\gamma_1, \gamma_2) \geq 6$, then $\gamma_1$ and $\gamma_2$ must intersect at least $\frac{(|\chi(S)|-2)^2}{2}$ times. Using the fact that $L_S \leq 2 \log(|\chi(S)|)+4$ for closed surfaces, it follows that $d_{C(S)}(\gamma_1, \gamma_2) \leq 5$ so long as $|\chi(S)| \geq 8$ since

$$x \geq 8 \Rightarrow \frac{(x-2)^2}{2} > 2 \cdot (2 \log(x) + 4).$$

For the finite list of remaining surfaces, we use the fact that on any surface $S$,

$$d_{C(S)}(\alpha, \beta) \leq 2 \log_2(i(\alpha, \beta)) + 2.$$

Note that if $|\chi(S)| < 8$, $i(\gamma_1, \gamma_2) \leq 2L_S < 16.32$, so we must have $d_{C(S)}(\gamma_1, \gamma_2) \leq 10$.

In general, let $\alpha$ represent the systole of $S$; $\gamma_1$ need not coincide with $\alpha$, but the above argument shows that

$$d_{C(S)}(\gamma_1, \alpha) \leq \begin{cases} 10 & |\chi(S)| < 8 \\ 5 & |\chi(S)| \geq 8 \end{cases}$$

and so by applying the same argument to $\gamma_2$ and then using the triangle inequality,

$$d_{C(S)}(\gamma_1, \gamma_2) \leq \begin{cases} 20 & |\chi(S)| < 8 \\ 10 & |\chi(S)| \geq 8 \end{cases}$$

The non-closed case:

As for the general case where $S$ is not necessarily closed, we use the collar lemma and argue that $\gamma_1, \gamma_2$ each have embedded collar neighborhoods of width at least

$$\log(\coth(L_S/4)),$$
which, applying (14), is at least
\[
\log \left( \coth \left( \frac{3}{2} \log(\pi |\chi(S)|) + \frac{7}{2} \right) \right).
\]
It follows that
\[
i(\gamma_1, \gamma_2) \leq \frac{6 \log(\pi |\chi(S)|) + 14}{\log \left( \coth \left( \frac{3}{2} \log(\pi |\chi(S)|) + \frac{7}{2} \right) \right)}.
\]
(18)
\[
= \frac{6 \log(\pi |\chi(S)|) + 14}{\log \left( \frac{e^{\pi^2 |\chi(S)|^2} + 1}{e^{\pi^2 |\chi(S)|^2} - 1} \right)} =: W_S.
\]
We compute directly that
\[
W_S \leq 240 \cdot |\chi(S)|^3 \log(|\chi(S)|).
\]
Assuming that $|\chi(S)| \geq 10$ and using (4), we conclude that
\[
d_{C(S)}(\gamma_1, \gamma_2) \leq 2 + 2 \cdot \frac{6 \log(2^{2n}|\chi(S)|^3 \log(|\chi(S)|))}{\log ((|\chi(S)| - 2)/2)}
\]
\[
\leq 2 + \frac{78}{\log(|\chi(S)| - 2) - 1} + \frac{8 \cdot \log |\chi(S)|}{\log(|\chi(S)| - 2) - 1} =: 2 + A + B.
\]
As we are assuming that $|\chi(S)| \geq 10$, this is in turn at most
\[
2 + 72.3 + 17.06 < 92.
\]
On the other hand, if $|\chi(S)| < 10$, $W_S$ is bounded from above and applying (3) yields
\[
d_{C(S)}(\gamma_1, \gamma_2) \leq 104.
\]
In conclusion,
\[
d_{C(S)}(\gamma_1, \gamma_2) \leq \begin{cases} 104 & |\chi(S)| < 10 \\ 92 & |\chi(S)| \geq 10 \end{cases}
\]
We note that as $|\chi(S)| \to \infty$, $A \to 0, B \to 8$ and thus for sufficiently large Euler characteristic, we obtain the much smaller bound of 11. Moreover, using a stronger version of (4) due to the first author [Aou13], one can conclude that for all $S$ with $|\chi(S)|$ sufficiently large,
\[
d_{C(S)}(\gamma_1, \gamma_2) < 6.
\]
\[\square\]

**Lemma 3.6.** Let $0 \leq \delta \leq 1$ and $L \geq 1$. Fix $x \in M_{[\delta, \infty)}$. Then the number of homotopy classes of loops of length less than $L$ based at $x$ is less than
\[
P(L, \delta) := \frac{\text{Vol}_3(L + \delta)}{\text{Vol}_3(\delta)}.
\]
Proposition 4.2. Let \( \delta \) denote the \( \delta \)-ball of radius \( \ell \) in \( M \) such that the polynomials \( \alpha \) is \( \ell \)-short. Otherwise, set tube \( \delta \)-electric distance between \( \alpha, \beta \) in \( M \) to be the shortest hyperbolic geodesic joining \( \alpha, \beta \). When \( M \) has no cusps, this is the length of the portion of the shortest hyperbolic geodesic joining \( \alpha, \beta \) that occurs outside of the \( \delta \)-tubes of \( M \). Our main technical result is an explicit inequality relating distance in the curve graph of \( S \) with electric distance in \( M \).

Theorem 4.1. Let \( \alpha, \beta \) be curves in \( S \) and let \( M \equiv S \times \mathbb{R} \) be a hyperbolic manifold such that \( \ell_M(\alpha), \ell_M(\beta) \leq \epsilon_3 \). Then
\[
1/A_1(|\chi(S)|) \cdot d_{C(S)}(\alpha, \beta) \leq d_{M}(\alpha, \beta) \leq A_2(|\chi(S)|) \cdot d_{C(S)}(\alpha, \beta),
\]
where the polynomials \( A_1 \) and \( A_2 \) are as in Equation (1).

The proof will be completed over the next several sections.

Fix \( 0 < \delta \leq \epsilon_3 \) and let \( \alpha \) be a curve in the hyperbolic manifold \( M \). Let tube\( \delta \) denote the \( \delta \)-tube \( T_\alpha(\delta) \) about the geodesic representative of \( \alpha \) in \( M \) in the case where \( \alpha \) is \( \delta \)-short. Otherwise, set tube\( \alpha \) to be the geodesic representative of \( \alpha \).

The idea behind the following proposition is simple and well-known to experts.

Proposition 4.2. Let \( 0 < \eta < \frac{\epsilon_3}{\chi(S)|^3} \). Then for any curves \( \alpha, \beta \) in \( S \),
\[
d_{M}(\alpha, \beta) \leq \frac{4\pi |\chi(S)| \cdot d_{C(S)}(\alpha, \beta)}{\eta}.
\]

Remark 4.3. When \( S \) is closed, \( \eta \) need only be less than the Margulis constant \( \epsilon_3 \). This fact will be used in the proof of Theorem 7.1.
Proof. Let $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ be a curve graph geodesic from $\alpha$ to $\beta$. For each $i$, let $f_i : X_i = (S, g_i) \to M$ be a pleated surface through $\alpha_i \cup \alpha_{i+1}$. In particular, $f_i$ maps each of the geodesic representatives of $\alpha_i$ and $\alpha_{i+1}$ in $X_i$ to its geodesic representative in $M$.

Let $p_i$ be the shortest path in $X_i$ from the geodesic representative of $\alpha_i$ to the geodesic representative of $\alpha_{i+1}$. The bounded diameter lemma of Thurston and Bonahon gives that length$(p_i \cap (X_i)_{[\eta, \infty)}) \leq \frac{4\pi|\chi(S)|}{\eta}$. Indeed, if $p_i^\eta = p_i \cap (X_i)_{[\eta, \infty)}$, then the $\eta/2$ neighborhood $C_i$ of $p_i^\eta$ is embedded in $X_i$ and so

$$\eta/2 \cdot \ell_{X_i}(p_i^\eta) \leq \text{Area}(C_i) \leq 2\pi|\chi(S)|.$$  

Since $f_i$ is 1-Lipschitz, it maps $\eta$-thin parts of $X_i$ to $\eta$-thin parts of $M$ and so length$(f_i(p_i) \cap M_{[\eta, \infty)}) \leq \ell_{X_i}(p_i^\eta)$. When $M$ has no cusps, this immediately gives that $d_M^\eta(\text{tube}_M(\alpha_i), \text{tube}_M(\alpha_{i+1})) \leq \ell_{X_i}(p_i^\eta)$.

In the presence of cusps, we argue as follows: First, we claim that $p_i$ cannot enter any horocyclic cusp neighborhood in $X_i$ whose boundary has length $2/e$. To see this, begin with the standard fact that simple closed geodesics on $X_i$ do not enter any standard cusp neighborhoods. So for any cusp of $X_i$, the endpoints of $p_i$ lie outside of its standard cusp neighborhood. Since $p_i$ is embedded, the length of any component of its intersection with a standard cusp neighborhood is no more than 2. Hence, its deepest point in the standard cusp neighborhood has distance no more than 1 from the horocycle boundary. This means that it does not cross the horocycle for that cusp of length $2/e$.

Now suppose that there is some $z \in p_i$ such that $f_i(z)$ lies in an $\eta$-cusp of $M$. Then any nontrivial loop based at $z$ whose length is less than $2\log(\epsilon_3/\eta)$ must be peripheral. This is because the image of such a loop is entirely contained in the corresponding $\epsilon_3$-cusp of $M$ and so the loop must represent a peripheral element of $\pi_1S$. But since

$$\eta < \frac{\epsilon_3}{e^6(\pi|\chi(S)|)^3},$$

we see that every loop of length no more than $6 \log(\pi|\chi(S)|) + 12$ based at $z$ is peripheral. However, the fact that $p_i$ does not enter any horocyclic cusp neighborhood in $X_i$ whose boundary has length $2/e$, together with Lemma 3.1 and Equation (14), implies that every point along $p_i$ is the basepoint of some essential (i.e. nonperipheral) loop of length no more than $6 \log(\pi|\chi(S)|) + 11$, a contradiction. Here we are using the fact that the injectivity radius along $p_i$ is at least $1/e$ so we set $\delta = 1/e$ in (14).

We conclude that $f_i(p_i)$ does not enter any $\eta$-cups of $M$. Hence, just as in the case without cusps, we conclude that $d_M^\eta(\text{tube}_M(\alpha_i), \text{tube}_M(\alpha_{i+1})) \leq \ell_{X_i}(p_i^\eta)$.
Finally, using the fact that $f_i$ maps the $\eta$-thin part of $X_i$ to the $\eta$-thin part of $M$, we obtain
\[
d_M^\alpha(\alpha, \beta) \leq \sum_{i=0}^{n-1} d_M^\alpha(\text{tube}_M(\alpha_i), \text{tube}_M(\alpha_{i+1})) \\
\leq \sum_{i=0}^{n-1} \ell_{X_i}(p_i) \\
\leq \frac{4\pi|\chi(S)|}{\eta} \cdot d_{\mathcal{C}(S)}(\alpha, \beta).
\]

We label the coefficient at the end of the proof above by
\[
(21) \quad A(x, \eta) = \frac{4\pi x}{\eta}.
\]
Thus, (21) and (16) yield the inequality
\[
(22) \quad A(|\chi(S)|, \epsilon_S) \leq A_2(|\chi(S)|),
\]
for
\[
(23) \quad A_2(x) = \begin{cases} 
4 e^{10 \pi x^3} & \text{for } S \text{ closed} \\
4 e^{20 \pi^7 x^7} & \text{otherwise},
\end{cases}
\]
as in Equation (1).

Hence, we can complete the proof of the upper bound in Theorem 4.1 using Proposition 4.2 with $\eta = \epsilon_S$ so that
\[
(24) \quad d_M^\epsilon(\alpha, \beta) \leq A(|\chi(S)|, \epsilon_S) \cdot d_{\mathcal{C}(S)}(\alpha, \beta) \\
\leq A_2(|\chi(S)|) \cdot d_{\mathcal{C}(S)}(\alpha, \beta)
\]
Note that for this upper bound there is no requirement on the lengths of $\alpha, \beta$.

The main idea for the other direction is contained in the following lemma. Roughly, the lemma says that as long as we can find a sweepout between the curves $\alpha$ and $\beta$ which separates $\alpha$ from $\beta$ at all times, then we obtain the desired bound on curve graph distance in terms of electric distance in $M$. The fact that we can find such a sweepout will be proved in the next section.

**Lemma 4.4.** Let $\alpha$ and $\beta$ be curves in $S$ and $M \cong S \times \mathbb{R}$ a hyperbolic manifold such that $\ell_M(\alpha), \ell_M(\beta) \leq \epsilon_S$. Let $p$ be a path in $M$ joining $\mathbb{T}_\alpha(\epsilon_S)$ and $\mathbb{T}_\beta(\epsilon_S)$ and suppose that
\begin{enumerate}
\item $p$ is contained in $M_{[\epsilon_S, \infty)}$,
\item there is a 1-Lipschitz sweepout $(f_t : X_t = (S, g_t) \to M)_{t \in [0,1]}$ such that $f_t(S) \cap p \neq \emptyset$ for all $t \in [0, 1]$, and
\item $f_0(S) \cap \mathbb{T}_\alpha(\epsilon_S) \neq \emptyset$ and $f_1(S) \cap \mathbb{T}_\beta(\epsilon_S) \neq \emptyset$.
\end{enumerate}
Then
\[ d_{C(S)}(\alpha, \beta) \leq A_1(|\chi(S)|) \cdot \ell_M(p) \]
for \( A_1(x) \) as in Equation (1).

**Proof.** Note that by Lemma 3.3, \( \ell_{X_a}(\alpha) \leq L_S \) and \( \ell_{X_b}(\beta) \leq L_S \). For each curve \( \gamma \) on \( S \), set
\[ I(\gamma) = \{ t \in [a, b] : \ell_{X_t}(\gamma_p) \leq L_S \}. \]
Here \( \gamma_p \) is the shortest loop over all representatives of \( \gamma \) on \( S \) with the property that \( f_t(\gamma_p) \) passes through the geodesic \( p \). By (2) (and Lemma 3.1), these intervals cover \([a, b]\).

Now break \( p \) up into \( N \) segments \( p_1, \ldots, p_N \), the first \( N - 1 \) of which have length 1 and the last of which has length less than 2 so that \( N = \lfloor \ell_M(p) \rfloor \). Let \( I_i \) be the set of all \( \gamma \) such that there is a \( t \in I(\gamma) \) with \( f_t(\gamma_p) \cap p_i \neq \emptyset \). That is, \( \gamma \) can be realized as a loop of length less than \( L_S \) starting at a point along \( p_i \). We note that it is possible that \( I_i \) is empty since we are not requiring that each point of \( p \) is in the image of a slice of the sweepout.

By criterion (1), Lemma 3.6, and Remark 3.7,
\[ \#I_i \leq \frac{\text{Vol}_3(L_S + \epsilon_S + 2)}{\text{Vol}_3(\epsilon_S)} \leq \frac{\sinh(2(L_S + \epsilon_S + 2))}{\epsilon_S^3} \leq \frac{e^{2(L_S+\epsilon_S+2)}}{\epsilon_S^3}. \]
Furthermore,
\[ e^{2(L_S+\epsilon_S+2)} \leq e^{2L_S+6} \leq e^6 \cdot e^{4\log(|\chi(S)|)+8} = e^{14|\chi(S)|^4}, \]
when \( S \) is closed, and
\[ e^{2(L_S+\epsilon_S+2)} \leq e^6 \cdot e^{12\log(\pi|\chi(S)|)+28} = e^{34\pi|\chi(S)|^{12}}, \]
when \( S \) has punctures. Along with the lower bounds on \( \epsilon_S \) established in (16) and (17) this implies,
\[ \#I_i \leq \begin{cases} e^{14|\chi(S)|^{10}} & \text{for } S \text{ closed} \\ e^{94\pi^{30}|\chi(S)|^{30}} & \text{otherwise.} \end{cases} \]
By Lemma 3.5, if \( I(\gamma_1) \cap I(\gamma_2) \neq \emptyset \), \( d_{C(S)}(\gamma_1, \gamma_2) \leq D \), where \( D \) is the universal constant from Lemma 3.5. Further, by the above paragraph, the total number of such curves seen along \( p \) is no more than \( \#I_i \). Since the sweepout passes through this set of curves with jumps of size no more than \( D \),
\[ d_{C(S)}(\alpha, \beta) \leq D \cdot \#I_i \cdot N \]
\[ \leq D \cdot \#I_i \cdot \ell_M(p) \]
By the proof of Lemma 3.5, $D \leq 20$ when $S$ is closed and $D \leq 104$ in general, so that setting

$$A_1(x) = \begin{cases} 
20e^{44}x^{10} & \text{for } S \text{ closed} \\
104e^{94}\pi^{30}x^{30} & \text{otherwise}
\end{cases}$$

as in Equation (1) gives us

$$d_{C(S)}(\alpha, \beta) \leq A_1(|\chi(S)|) \cdot \ell_M(p).$$

\[\Box\]

### 5. Separating sweepouts

In what follows, let $T_\alpha$ be shorthand for the tube $T_\alpha(\epsilon_S)$. In order to find sweepouts satisfying the conditions of Lemma 4.4, we require the following:

**Proposition 5.1.** Let $\alpha, \beta$ be intersecting curves on $S$ whose lengths in $M$ are no more than $\epsilon_S$. Let $f_t: S \rightarrow M, t \in [a, b]$ be a 1-Lipschitz sweepout such that $\Sigma_a$ lies to the left of $\alpha, \beta$ and $\Sigma_b$ lies to the right of $\alpha, \beta$, where $\Sigma_\ell = f_t(S)$. Then there is a subinterval $[c, d] \subset [a, b]$ such that

1. Both $T_\alpha$ and $T_\beta$ meet $\Sigma \cup \Sigma_d$.
2. Neither $T_\alpha$ nor $T_\beta$ meet $\Sigma_t$ for $t \in (c, d)$, and
3. $\Sigma_t$ separates $T_\alpha$ from $T_\beta$ for each $t \in (c, d)$.

The proof requires some notation. Let $m_\alpha \subset [a, b]$ be the set of times the sweepout meets $T_\alpha$:

$$m_\alpha = \{t \in [a, b] : \Sigma_t \cap T_\alpha \neq \emptyset\}.$$  

Define $m_\beta$ similarly, and note that $m_\alpha$ and $m_\beta$ are disjoint closed subsets of $[a, b]$, since no 1-Lipschitz map can meet both $T_\alpha$ and $T_\beta$. This follows from the fact that if $\Sigma_\ell$ meets both $T_\alpha$ and $T_\beta$, then by Lemma 3.3 there are representative loops $a$ and $b$ on $S$ such that $f_t(a) \subset T_\alpha$ and $f_t(b) \subset T_\beta$, and so $a$ and $b$ are disjoint. This contradicts the assumption that $\alpha$ and $\beta$ intersect.

The components of $[a, b]\setminus m_\alpha$ are open in the interval $[a, b]$, and each is a subset of one of three disjoint subsets of $[a, b]$, denoted $l_\alpha, r_\alpha, b_\alpha$ and defined as follows. By definition $l_\alpha$ consists of those times when $\Sigma_t$ is to the left of $T_\alpha$. This means that $T_\alpha$ lies in the component of $M\setminus \Sigma_t$ containing the $\lambda^+$ end of $M$. Similarly, let $r_\alpha$ be those times for which $\Sigma_t$ lies to the right of $T_\alpha$, and let $b_\alpha$ be those time when $T_\alpha$ lies in a bounded component of $M\setminus \Sigma_t$. Since $\Sigma_t$ always separates $M$, $[a, b]\setminus m_\alpha = l_\alpha \cup b_\alpha \cup r_\alpha$.

Define $l_\beta, b_\beta, r_\beta$ in the analogous way, and note that $a \in l_\alpha \cap l_\beta$ and $b \in r_\alpha \cap r_\beta$ by hypothesis. We will think of each point in $[a, b]$ as being colored by the subsets they are in - each point gets an $\alpha$ color and a $\beta$ color.  

With this terminology, we claim that the following lemma immediately proves Proposition 5.1.

**Lemma 5.2.** There is a closed interval $I \subset [a, b]$ whose interior is a component of $[a, b]\setminus (m_\alpha \cup m_\beta)$ such that

1. $I$ has one endpoint in $m_\alpha$ and one endpoint in $m_\beta$, and
(2) for each \( t \) in the interior of \( I \), its \( \alpha \) color is different from its \( \beta \) color.

Proposition 5.1 follows from the fact that if \( t \) gets a different \( \alpha \) color and \( \beta \) color (the colors being either \( l, r, b \)) then \( \mathcal{T}_\alpha \) and \( \mathcal{T}_\beta \) lie in different components of \( M \setminus \Sigma_t \).

We now turn to finding the desired subinterval of \([a, b] \). Let us begin by making a few observations. First, \( m_\alpha \) and \( m_\beta \) are closed and disjoint, so components of one cannot accumulate onto a component of the other. Hence, if we are at a component of (say) \( m_\alpha \) it makes sense to talk about the component of \( m_\beta \) immediately after or before it in the time interval. Similarly, any monotone (with respect to the order on \([a, b] \)) sequence of components of \( m_\alpha \cup m_\beta \) alternating between components of \( m_\alpha \) and \( m_\beta \) must terminate after finitely many steps. Second, in what follows we only consider components of \( m_\alpha \cup m_\beta \) which have nonempty interior. We call such components thick. Note that by continuity of the sweepout, the \( \alpha \) color can change only across a thick \( m_\alpha \) component. More accurately, if two points in \([a, b] \) are not separated by a thick component of \( m_\alpha \), then they have the same \( \alpha \) color. Finally, call an interval in \([a, b] \backslash (m_\alpha \cup m_\beta) \) switching if it has one endpoint in \( m_\alpha \) and one endpoint in \( m_\beta \). It is clear that a switching interval must exist: otherwise we can construct a sequence of nested intervals \( I_0 \supset I_1 \supset \ldots \) each with one endpoint in \( m_\alpha \) and one endpoint in \( m_\beta \) such that \( \cap I_k = \{x\} \). Since we would necessarily have that \( x \in m_\alpha \cap m_\beta \), this is a contradiction.

Proof of Lemma 5.2. By an \( m_\alpha \) component we mean a thick connected component of \( m_\alpha \), and the same goes for \( \beta \). Let us look at some time in the interval \([a, b] \) that we see an \( m_\alpha \) component followed by an \( m_\beta \) component or vice versa. This exists since there is a switching interval. Up to reversing the time parameter, we assume that the \( m_\alpha \) component comes first. Let \( m_{\alpha_0} \) denote that \( m_\alpha \) component, and \( m_{\beta_0} \) the \( m_\beta \) component.

Let \( m_{\alpha_1} \) be the first \( m_\alpha \) component after \( m_{\beta_0} \) and let blue be the \( \alpha \)-color of the interval in between \( m_{\alpha_0} \) and \( m_{\alpha_1} \). We assume that the \( \beta \)-color to the left of \( m_{\beta_0} \) is also blue, otherwise the interval we are looking for is \([m_{\alpha_0}, m_{\beta_0}] \). (This notation means the interval between the endpoints of these components not containing their interior.)

Let \( m_{\beta_1} \) be the last \( m_\beta \) component before \( m_{\alpha_1} \) and let \( m_{\beta_2} \) be the next \( m_\beta \) component after \( m_{\beta_1} \) (and thus the first one after \( m_{\alpha_1} \), so it is well-defined). If the interval between \( m_{\beta_1} \) and \( m_{\beta_2} \) is not blue, say it is green, then the interval we are looking for is the one between \( m_{\beta_1} \) and \( m_{\alpha_1} \). It is colored \( \alpha \)-blue but \( \beta \)-green. Thus, we assume for contradiction that the interval between \( m_{\beta_1} \) and \( m_{\beta_2} \) is in fact blue (see Figure 2).

We will in general argue that if we have not found our desired time interval, and we see two \( m_\alpha \) components, \( m'_\alpha \) and \( m''_\alpha \) with the color of the interval \([m'_\alpha, m''_\alpha] \) colored blue, then the color to the left of the \( m_\beta \) component directly after \( m''_\alpha \) is also blue.

Now let \( m_{\alpha_2} \) and \( m_{\alpha_3} \) be the \( \alpha \) component directly before and after \( m_{\beta_2} \). If \([m_{\alpha_2}, m_{\alpha_3}] \) is not blue, say it is green, then the interval we are looking for is \([m_{\alpha_2}, m_{\beta_3}] \); it is \( \alpha \)-green but \( \beta \)-blue (see Figure 3). Assume that it is blue, and let \( m_{\beta_3} \) and \( m_{\beta_4} \) be the \( m_\beta \) components directly before and after \( m_{\alpha_3} \), respectively. If \([m_{\beta_3}, m_{\beta_4}] \) is not blue, then we are again done. The interval we want is \([m_{\beta_3}, m_{\alpha_3}] \). Thus we can assume that both are blue, and we have succeeded in demonstrating the set up outlined in the previous
Figure 2. We represent the time interval \([a, b]\) as two separate intervals to keep track of relevant \(\alpha\) and \(\beta\) information.

Paragraph since the color to the left of \(m_{\beta_4}\) is blue (see Figure 4). This perpetuates to the right.

Figure 3.

Figure 4.

Recall that the last \(\alpha\) interval and the last \(\beta\) interval must be the same color (these are the intervals ending in \(b\)). Suppose that the last \(\alpha\) interval and the last \(\beta\) interval are not blue, say they are red. We now show that, in this case, we will always find the interval we are looking for.
Let \( m_{\beta_n} \) be the last \( m_{\beta} \) component before \( b \) resulting from the argument outlined above; \( m_{\beta_n} \) exists otherwise there would be an infinite alternating sequence of \( m_{\alpha} \) and \( m_{\beta} \) components, which we said above cannot happen. Let \( m_{\alpha_n} \) be the \( m_{\alpha} \) component directly before \( m_{\beta_n} \), and note that the color to the left of \( m_{\beta_n} \) is blue. We claim that \( m_{\beta_n} \) is the last \( m_{\beta} \) component, or \( m_{\alpha_n} \) is the last \( m_{\alpha} \) component. Otherwise the argument outlined above applies again, and we either find our desired interval or we have obtained a contradiction to the choice of \( m_{\beta_n} \).

Suppose that \( m_{\beta_n} \) is the last \( m_{\beta} \) component of the interval \([a, b]\) and that \( m_{\alpha_n} \) is not the last \( m_{\alpha} \) component. Let \( m_{\alpha_{n+1}} \) be the \( m_{\alpha} \) component directly after \( m_{\beta_n} \) and consider the interval \([m_{\alpha_n}, m_{\alpha_{n+1}}]\). If it is blue, the interval we are looking for is \([m_{\beta_n}, m_{\alpha_{n+1}}]\) (see Figure 5) and if it is not, then the interval we are looking for is \([m_{\alpha_n}, m_{\beta_n}]\).

\[ \text{Figure 5.} \]

Next suppose that \( m_{\alpha_n} \) is the last \( m_{\alpha} \) component. Then by our hypothesis, the color to the right of \( m_{\alpha_n} \) is red. Thus, the interval we are looking for is \([m_{\alpha_n}, m_{\beta_n}]\) which is \( \alpha \)-red but \( \beta \)-blue (see Figure 6).

\[ \text{Figure 6.} \]

So we assume for now that the last \( \alpha \)- and \( \beta \)- intervals are blue. A similar perpetuation argument applies as we move left in the interval \([a, b]\) so that either we find our desired interval, or the first \( \alpha \)- and \( \beta \)- intervals, which begin at \( a \), are also blue. This contradicts the fact that the sweepout starts with \( \Sigma_a \) to the left of \( \alpha \) and \( \beta \), and ends with \( \Sigma_b \) to the right of \( \alpha \) and \( \beta \). \( \square \)
Another method for proving Lemma 5.1 was suggested to the authors by Dave Futer. In short, one uses a result of Otal [Ota95, Ota03], which guarantees that short curves in $M$ are unlinked, to topologically order the short $\gamma_i$ and the 1-Lipschitz surfaces they meet. Rather than attempt to make effective this technique, we chose instead to employ the direct combinatorial argument found here.

6. Finishing the proof of Theorem 4.1

Recall that $A_2(\lvert \chi(S) \rvert)$ is obtained by setting $\eta = \epsilon_S$ in (21), giving us the upper bound in Theorem 4.1

$$d_M^s(\alpha, \beta) \leq \frac{2\pi |\chi(S)|}{\epsilon_S} \cdot d_C(S)(\alpha, \beta) = A_2(\lvert \chi(S) \rvert) \cdot d_C(S)(\alpha, \beta).$$

For the lower bound, suppose that $\alpha$ and $\beta$ are given and let $p$ be a geodesic from $\alpha$ to $\beta$ in $M$. Let $S$ be the set of curves $\gamma$ in $S$ such that $p$ meets $T_\gamma = T_\gamma(\epsilon_S)$ in $M$ and index $S = \{\gamma_t\}_{t=1}^N$ according to the order in which these tubes are met by $p$. (Set $\gamma_0 = \alpha$ and $\gamma_{N+1} = \beta$.)

Let $p_i$ be the subarc of $p$ between the last point of $p \cap T_\gamma$ and the first point of $p \cap T_{\gamma_{i+1}}$. Then these subarcs are disjoint and $p \cap M_{[\epsilon_S, \infty)} = \bigcup_i p_i$. Additionally, let $f_t, t \in [a, b]$ be a sweepout of 1-Lipschitz maps such that $\Sigma_a$ lies to the left of $\alpha, \beta, \gamma_i$ and $\Sigma_b$ lies to the right of $\alpha, \beta, \gamma_i$ for $1 \leq i \leq N$. Such a sweepout always exists by Theorem 2.1.

Our analysis breaks into two cases, depending on whether $\gamma_i$ and $\gamma_{i+1}$ intersect as curves on $S$. If not, then $d_C(S)(\gamma_i, \gamma_{i+1}) \leq 1$ and we can only say that $\ell_M(p_i) \geq 4$ by Remark 3.4.

Now suppose that $\gamma_i$ and $\gamma_{i+1}$ are such that $d_C(S)(\gamma_i, \gamma_{i+1}) > 2$. In this case, apply Proposition 5.1 to obtain (up to reversing the time parameter) a sub-sweepout $(f_t : X_t = (S, g_t) \to M)_{t \in [a, b]}$ with the following properties:

1. $\Sigma_a \cap T_\gamma \neq \emptyset$ and $\Sigma_b \cap T_{\gamma_{i+1}} \neq \emptyset$,
2. $\Sigma_t \cap p_i \neq \emptyset$ for all $t \in [a, b]$,
3. $p_i$ is contained in $M_{[\epsilon_S, \infty)}$.

Note that we are using that $\Sigma_t$ cannot meet both $T_\gamma$ and $T_{\gamma_{i+1}}$ and that since $\Sigma_t$ separates $\gamma_i$ from $\gamma_{i+1}$, it must meet $p_i$. Hence, we may apply Lemma 4.4 to conclude that

$$d_C(S)(\gamma_i, \gamma_{i+1}) \leq A_1(\lvert \chi(S) \rvert) \cdot \ell_M(p_i),$$

and thus,

$$d_C(S)(\alpha, \beta) \leq \sum_i d_C(S)(\gamma_i, \gamma_{i+1}) \leq A_1(\lvert \chi(S) \rvert) \cdot \sum_i \ell_M(p_i) \leq A_1(\lvert \chi(S) \rvert) \cdot d_M^s(\alpha, \beta)$$

as wanted. This completes the proof of Theorem 4.1.
7. Covers and the curve complex

In this section, we follow Tang [Tan12] and apply Theorem 4.1 to analyze maps between curve graphs induced by covering maps of surfaces.

If \( p : \tilde{S} \to S \) is a covering map, there is a coarsely well-defined map \( p^* : \mathcal{C}(S) \to \mathcal{C}(\tilde{S}) \) induced by \( p \); given an essential simple closed curve \( \gamma \) on \( S \), define \( p^*(\gamma) \) to be the full pre-image \( \tilde{S}^1(\gamma) \subseteq \tilde{S} \). This will be a multi-curve on \( \tilde{S} \) corresponding to a complete subgraph of \( \mathcal{C}(\tilde{S}) \). Given \( \alpha \) and \( \beta \) vertices of \( \mathcal{C}(S) \), we can then define the distance in \( \mathcal{C}(\tilde{S}) \) between \( p^*(\alpha) \) and \( p^*(\beta) \) to be the diameter of their union:

\[
d_{\mathcal{C}(\tilde{S})}(p^*(\alpha), p^*(\beta)) := \text{diam}(p^*(\alpha) \cup p^*(\beta)).
\]

With this setup, we prove the following:

**Theorem 7.1.** Let \( p : \tilde{S} \to S \) be a finite covering map between non-sporadic surfaces \( \tilde{S}, S \). Then for any \( \alpha, \beta \) distinct essential simple closed curves on \( S \),

\[
\frac{d_{\mathcal{C}(S)}(\alpha, \beta)}{\deg(p) \cdot A_3(|\chi(S)|)} \leq d_{\mathcal{C}(\tilde{S})}(p^*(\alpha), p^*(\beta)) \leq d_{\mathcal{C}(S)}(\alpha, \beta),
\]

where \( A_3 \) is the polynomial \( A_3(x) = 80 e^{54} \pi x^{13} \) when \( S \) is closed.

When \( S \) has punctures,

\[
\frac{d_{\mathcal{C}(S)}(\alpha, \beta)}{\deg(p)^4 \cdot A_3(|\chi(S)|)} \leq d_{\mathcal{C}(\tilde{S})}(p^*(\alpha), p^*(\beta)) \leq d_{\mathcal{C}(S)}(\alpha, \beta),
\]

where \( A_3 \) is the polynomial \( A_3(x) = 416 e^{114} \pi x^{37} \).

Recall that \( A_3(x) = A_1(x) \cdot A_2(x) \) for \( A_1, A_2 \) as in Equation 1

*Proof.* Given \( \gamma_1, \gamma_2 \) disjoint essential simple closed curves on \( S \), \( p^*(\gamma_1) \) will be disjoint from \( p^*(\gamma_2) \). This proves the upper bound on \( d_{\mathcal{C}(\tilde{S})}(p^*(\alpha), p^*(\beta)) \) in Theorem 7.1.

For the lower bound, we choose a hyperbolic manifold \( M \cong S \times \mathbb{R} \) so that \( \ell_M(\alpha) \) and \( \ell_M(\beta) \) are at most \( \epsilon_S \). Constructing such a manifold is standard, see [Kap10, Chapter 8]. Thus, the first inequality of Theorem 4.1 implies that

\[
d_{\mathcal{C}(S)}(\alpha, \beta) \leq A_1(|\chi(S)|) \cdot d_M^{\epsilon}(\alpha, \beta).
\]

The covering map \( p \) gives rise to a covering of 3-manifolds between \( p^*M \) and \( M \). Let \( p^*\alpha, p^*\beta \) also denote the geodesic representatives in \( p^*M \) of the lifts \( p^{-1}(\alpha), p^{-1}(\beta) \), respectively, and let \( \gamma \) be a path in \( p^*M \) from any component of \( p^*\alpha \) to any component of \( p^*\beta \). Then \( \gamma \) maps to a path in \( M \) from \( \alpha \) to \( \beta \).

Since a covering map is distance non-increasing and sends the thin part into the thin part, it follows that

\[
d_M^{\epsilon}(\alpha, \beta) \leq d_{p^*M}^{\epsilon}(p^*\alpha, p^*\beta),
\]

where the left hand side is defined to be the minimum electric distance between a tube about any component of \( p^*\alpha \) and a tube about any component of \( p^*\beta \). Combining this observation with (26) yields

\[
d_{\mathcal{C}(S)}(\alpha, \beta) \leq A_1(|\chi(S)|) \cdot d_{p^*M}^{\epsilon}(p^*\alpha, p^*\beta).
\]
When $S$ is closed, the upper bound on $\eta$ in Proposition 4.2 is simply $\epsilon_3$, the Margulis constant. Thus, applying Proposition 4.2 to the right hand side of (27) with $\eta = \epsilon_S$ we obtain
\[ d_{C(S)}(\alpha, \beta) \leq A_1(\|\chi(S)\|) \cdot A(\|\chi(\tilde{S})\|, \epsilon_S) \cdot d_{C(\tilde{S})}(p^*\alpha, p^*\beta). \]
Recall that $A(\|\chi(\tilde{S})\|, \epsilon_S) = \deg(p) \cdot A(\|\chi(S)\|, \epsilon_S) \leq \deg(p) \cdot A_2(\|\chi(S)\|)$ by (22), which yields the lower bound in the closed case.

In order to apply Proposition 4.2 to the right hand side of (27) when $S$ is not closed, it is necessary to choose
\[ \eta < \frac{\epsilon_3}{e^6(\pi|\chi(S)|)^3}; \]
and when the degree of $p: \tilde{S} \to S$ is large we note that $\epsilon_S$ is not small enough to satisfy the above inequality. Hence, we set $\eta = \min\{\epsilon_S, \frac{\epsilon_3}{e^6(\pi|\chi(S)|)^3}\}$. We first note that since $\eta \leq \epsilon_S$,
\[ d^{\epsilon_S}_{p^* M}(p^*\alpha, p^*\beta) \leq d^n_{p^* M}(p^*\alpha, p^*\beta). \]
Now starting with (27) and applying Proposition 4.2 with this $\eta$ yields
\[ d_{C(S)}(\alpha, \beta) \leq A_1(\|\chi(S)\|) \cdot A(\|\chi(\tilde{S})\|, \eta) \cdot d_{C(\tilde{S})}(p^*\alpha, p^*\beta), \]
where
\[ A(\|\chi(\tilde{S})\|, \eta) \leq \max\{\deg(p) \cdot A_2(\|\chi(S)\|), \deg(p)^4 \cdot 4e^9\pi^4(\|\chi(S)\|^4)\}, \]
which in either case is less than $\deg(p)^4 \cdot A_2(\|\chi(S)\|)$. This completes the proof. \(\square\)

Corollary 1.1 is immediate from Theorem 7.1 after noting that if
\[ d_{C(\tilde{S})}(p^*\alpha, p^*\beta) \geq 4, \]
then every lift of $\alpha$ intersects every lift of $\beta$.

**Remark 7.2.** We conclude this section by showing that the linear dependence on $\deg(p)$ in Theorem 7.1 is sharp in the closed case.

Let $S = S_\varrho$ ($\varrho \geq 2$) be a fixed closed surface with curves $\alpha$ and $\beta$ such that $\alpha$ is nonseparating, $\beta$ is separating, and $\alpha$ and $\beta$ fill $S$. Let $I = i(\alpha, \beta)$ and let $\tilde{S}_n$ be the $n$-fold cyclic cover of $S$ built as follows: Take $n$ copies of $X = S \setminus \alpha$, $X_0 \ldots X_{n-1}$ and glue them cyclically along their boundaries. That is, if we let $\tilde{\partial}X = \alpha^I \cup \alpha^\tau$, then we glue $\alpha_i^\tau$ to $\alpha_{i+1}^\tau$ for each $i \mod n$. Rename the resulting curve $\alpha_i^\tau = \alpha_{i+1}^\tau$ by $\tilde{\alpha}_i$. These are the preimages of $\alpha$ in $\tilde{S}_n$.

We note that $\tilde{S}_n$ is the cover corresponding to the kernel of the homomorphism $\phi: \pi_1(S) \to \mathbb{Z}/n\mathbb{Z}$ taking a loop to its algebraic intersection number with $\alpha$, mod $n$. Hence, $\beta$ has $n$ lifts to the cover $\tilde{S}_n$ and any such lift intersects no more than $I$ of the $\tilde{\alpha}_i$. In particular, each lift of $\beta$ lives in $X_i \cup X_{i+1} \cup \ldots \cup X_{i+t}$ for some $0 \leq i \leq n$.

Now set $f = \tau_\beta^{-1}\tau_\alpha$, which is pseudo-Anosov by Thurston’s criterion ([Thu88]). Hence, there is a $\kappa > 0$ (depending only on $S$) such that $d_{C(S)}(\alpha, f^j(\alpha)) \geq \kappa j$ ([MM99], [GT11]). Since both $\tau_\alpha$ and $\tau_\beta$ fix the homology class of $\alpha$ (recall that $\beta$ is a separating curve), so does $f$. Thus, $f$ fixes the kernel of $\phi$ and hence lifts to a map $\tilde{f}: \tilde{S}_n \to \tilde{S}_n$.  

Indeed, if we denote by $\tilde{\alpha}$ and $\tilde{\beta}$ the full preimage of $\alpha$ and $\beta$, one such lift is $\tilde{f} = \tau^{-1}_\beta \tau_\alpha$, a composition of multitwists.

But then, we must have

$$\tilde{f}(\tilde{\alpha}_1) = \tau^{-1}_\beta(\tilde{\alpha}_1) = \tau^{-1}_\beta(\tilde{\alpha}_1),$$

where $\tilde{\beta}$ is a multicurve consisting of components of $\tilde{\beta}$ that meet $\tilde{\alpha}_1$. Hence, from our observation above, $\tilde{\beta}$, and therefore $\tilde{f}(\tilde{\alpha}_1)$, is supported in $X_{-1} \cup X_{-1+1} \cup \ldots \cup X_I$.

Therefore, so long as $n \geq 2I + 1$, we have that $d_{C(\tilde{S}_n)}(\tilde{\alpha}_1, \tilde{f}(\tilde{\alpha}_1)) \leq 2$.

This construction can be iterated by choosing $n \geq 2jI + 1$, and considering $\tilde{f}^j(\tilde{\alpha}_1)$

This curve is contained in $Y_j := X_{-jI} \cup X_{j+1} \cup \ldots \cup X_{jI}$, which is a proper subsurface of $\tilde{S}_n$. Indeed the Euler characteristic of $Y_j$ is $2j \chi(S)$, which, in absolute value, is strictly less than $|\chi(\tilde{S}_n)|$ under the assumption that $n \geq 2jI + 1$. Hence, $d_{C(\tilde{S}_n)}(\tilde{\alpha}_1, \tilde{f}^j(\tilde{\alpha}_1)) \leq 2$ for $n \geq 2jI + 1$.

Now since $\tilde{f}^j(\tilde{\alpha}_1) \subset \tilde{f}^j(\tilde{\alpha}) = \tilde{f}^j(\alpha)$, we can set $\gamma_j = f^j(\alpha)$ to see that we have produced curves $\alpha$ and $\gamma_j$ on $S$ that have distance at least $\kappa j$ and a degree $n = 2jI + 1$ cover $\tilde{S}_n$ such that $d_{C(\tilde{S}_n)}(\tilde{\alpha}, \tilde{\gamma}_j) \leq 2$.

8. Application to Quantified Virtual Specialness

In this section we give an application of Theorem 7.1 to dual cube complexes for collections of curves on closed surfaces and their special covers.

8.1. Dual cube complexes and Sageev’s construction. Given a finite and filling collection $\Gamma$ of simple closed curves on a closed surface $S$, Sageev’s construction [Sag95] gives rise to a dual CAT(0) cube complex $\tilde{\mathcal{C}}_{\Gamma}$, on which $\pi_1 S$ acts freely, properly discontinuously, and cocompactly. The quotient of $\tilde{\mathcal{C}}_{\Gamma}$ by this action is a non-positively curved cube complex $\mathcal{C}_{\Gamma}$, which can be thought of as a cubulation of the surface $S$ since $\pi_1 S \cong \pi_1 \mathcal{C}_{\Gamma}$.

The construction of $\tilde{\mathcal{C}}_{\Gamma}$ roughly goes as follows. In the language of Wise [Wis00], the full preimage $\tilde{\Gamma}$ of $\Gamma$ in the universal cover $\tilde{S}$ of $S$ is a union of elevations, which each split $\tilde{S}$ into two half-spaces. A labelling of $\tilde{\Gamma}$ is a choice of half-space for each elevation in $\tilde{\Gamma}$, and the admissible labelings form the vertex set for $\tilde{\mathcal{C}}_{\Gamma}$. (For more details on admissible labellings see [Sag95].) Two labellings are joined by an edge when they differ on the choice of a half-space for exactly one elevation. The unique CAT(0) cube complex defined by this 1-skeleton is $\tilde{\mathcal{C}}_{\Gamma}$, and there is an intersection preserving identification of the curves in the system $\Gamma$ with the hyperplanes of $\tilde{\mathcal{C}}_{\Gamma}$. The action of $\pi_1 S$ on $\tilde{S}$ permutes the elevations, inducing an isometry of $\tilde{\mathcal{C}}_{\Gamma}$. We note that this construction of cube complexes works in a far more general setting. We summarize Sageev’s construction with the following theorem:
Theorem 8.1 (Sageev). Suppose $\Gamma$ is a finite, filling collection of curves on $S$. Then the dual cube complex $\overline{\mathcal{C}}_\Gamma$ is CAT(0) and there is an intersection preserving identification of the curves in $\Gamma$ with the hyperplanes of $\overline{\mathcal{C}}_\Gamma$. The group $\pi_1 S$ acts freely, properly discontinuously, and cocompactly on $\overline{\mathcal{C}}_\Gamma$.

8.2. Virtual specialness. It is well known that there exists a finite cover $\overline{\mathcal{C}}_\Gamma$ of $\mathcal{C}_\Gamma$ which is special [HW08]. Here $\overline{\mathcal{C}}_\Gamma$ is called special because its hyperplanes avoid three key pathologies (self-intersecton, direct self-osculation, and inter-osculation). There is an algebraic characterization of specialness [HW08]: that $\pi_1 \overline{\mathcal{C}}_\Gamma$ embeds in a particular right-angled Artin group (RAAG). The defining graph of that RAAG is the crossing graph of $\overline{\mathcal{C}}_\Gamma$. The crossing graph of $\overline{\mathcal{C}}_\Gamma$ is the simplicial graph whose vertices are hyperplanes of $\overline{\mathcal{C}}_\Gamma$ and whose edges connect distinct, intersecting hyperplanes.

Theorem 8.1 implies that the specialness of a cube complex dual to a collection of curves on a surface is determined by the intersection pattern of the underlying curves.

Suppose that $\Gamma$ consists of two simple closed curves, $\alpha$ and $\beta$, with nontrivial geometric self-intersection number and that together fill the surface $S$. Consider a finite-degree covering map $p : \tilde{S} \to S$, and as in Section 7 let $p^* : \mathcal{C}(S) \to \mathcal{C}(\tilde{S})$ be the induced map between their curve complexes.

There is also an induced covering map on the level of dual cube complexes $p_* : \mathcal{C}_{\Gamma'} \to \mathcal{C}_\Gamma$ where $\mathcal{C}_{\Gamma'}$ is the dual complex to the curve system $\Gamma' = p^{-1}(\alpha) \cup p^{-1}(\beta)$ on $\tilde{S}$ and is also the cover of $\mathcal{C}_\Gamma$ corresponding to the subgroup $\pi_1 \tilde{S} < \pi_1 S \cong \pi_1 \mathcal{C}_\Gamma$. We record the following lemma as an obstruction to the specialness of $\mathcal{C}_{\Gamma'}$.

Lemma 8.2. Suppose that $\alpha$ and $\beta$ are two simple closed curves that nontrivially intersect and together fill a surface $S$, and that $p : \tilde{S} \to S$ is a finite degree covering map. If every lift of $\alpha$ to $\tilde{S}$ intersects every lift of $\beta$ to $\tilde{S}$, then the cover $\mathcal{C}_{\Gamma'}$ of $\mathcal{C}_\Gamma$ corresponding to $\pi_1 \tilde{S} < \pi_1 S \cong \pi_1 \mathcal{C}_\Gamma$ cannot be special.

Proof. If every lift of $\alpha$ intersects every lift of $\beta$, then the underlying graph for the right-angled Artin group $A$ in which $\mathcal{C}_{\Gamma'}$ should embed is the join of two sets of non-adjacent vertices. Thus, $A = F_n \times F_m$ is the product of two free groups. However, $\pi_1 \mathcal{C}_{\Gamma'}$ is a surface group, which cannot embed in the product of two free groups [BR84].

Note that if $d_{\mathcal{C}(\tilde{S})}(p^* \alpha, p^* \beta) \geq 4$, then every lift of $\alpha$ intersects every lift of $\beta$. Thus, Theorem 7.1 gives us the following:

Theorem 8.3. Suppose that $\alpha$ and $\beta$ are two simple closed curves that together fill a closed surface $S$. Let $\deg \mathcal{C}_\Gamma$ be the minimal degree of a special cover of the dual cube complex $\mathcal{C}_\Gamma$ to the curve system $\Gamma = \alpha \cup \beta$. Then

$$\frac{d_{\mathcal{C}(\tilde{S})}(\alpha, \beta)}{C(S)} \leq \deg \mathcal{C}_\Gamma,$$

where $C(S)$ is a polynomial in $|\chi(S)|$ of degree 13.

Proof. Suppose that $p : \tilde{S} \to S$ is a finite degree cover of the surface $S$ and that $p_* : \mathcal{C}_{\Gamma'} \to \mathcal{C}_\Gamma$ is the induced cover of cube complexes. Additionally, assume that $\mathcal{C}_{\Gamma'}$ is
special. Theorem 7.1 gives us that
\[ \frac{d_{C(S)}(\alpha, \beta)}{\deg(p) \cdot A_1(|\chi(S)|) \cdot A_2(|\chi(S)|)} \leq d_{C(S)}(p^*(\alpha), p^*(\beta)). \]
Given that \( S \) is closed, \( A_1(|\chi(S)|) \) is a polynomial of degree 10 in \( |\chi(S)| \) and \( A_2(|\chi(S)|) \) is a polynomial of degree 3 in \( |\chi(S)| \) (see Equation (1)). Lemma 8.2 shows that \( \mathcal{C}_v \) cannot be special unless \( d_{C(S)}(p^*(\alpha), p^*(\beta)) \leq 3 \). Combining these results and solving for \( \deg(p) \) gives
\[ \frac{d_{C(S)}(\alpha, \beta)}{C(S)} \leq \deg(p), \]
where \( C(S) = 3 \cdot A_1(|\chi(S)|) \cdot A_2(|\chi(S)|) \).

9. The circumference of a fibered manifold

The methods developed above generalize to effectively relate the electric circumference of a fibered manifold to the curve graph translation length of its monodromy. The noneffective version of this relation has proven useful, for example, in work of Biringer–Souto on the rank of the fundamental group of such manifolds [BS15]. As in the previous section, we restrict to the case where \( S \) is closed.

Let \( \phi \in \text{Mod}(S) \) be pseudo-Anosov and denote its mapping torus by \( M_\phi \). For \( 0 < \delta < \varepsilon_3 \), denote the hyperbolic circumference and \( \delta \)-electric circumference of \( M_\phi \) by \( \text{circ}(M_\phi) \) and \( \text{circ}_\delta(M_\phi) \), respectively. That is, \( \text{circ}(M_\phi) \) is the minimum geodesic length of a loop in \( M \) which is not in the kernel of the associated map \( \pi_1(M_\phi) \to \mathbb{Z} \), and similarly \( \text{circ}_\delta(M_\phi) \) is the minimum \( \delta \)-electric length of a loop in \( M \) which is not in the kernel the map. Let \( \ell_S(\phi) \) be the stable translation length of \( \phi \) in \( C(S) \),
\[ \ell_S(\phi) = \lim_{n \to \infty} \frac{d_{C(S)}(\alpha, \phi^n \alpha)}{n}. \]

**Theorem 9.1.** If \( \phi : S \to S \) is a pseudo-Anosov homeomorphism of a closed surface \( S \), then
\[ \frac{1}{A_1(|\chi(S)|)} \cdot \ell_S(\phi) \leq \text{circ}_\delta(M_\phi) \leq A_2(|\chi(S)|) \cdot (\ell_S(\phi) + 2), \]
where the polynomials \( A_1 \) and \( A_2 \) are as in Equation (1).

Our argument follows the outline from Brock in [Bro03b]. There, Brock extends his theorem on volumes of quasi-fuchsian manifolds to volumes of hyperbolic mapping tori. Similarly, we deduce Theorem 9.1 from the tools we used to prove Theorem 4.1.

**Proof.** Let \( M = M_\phi \) and let \( N \) be the infinite cyclic cover of \( M \) corresponding to \( S \), The inclusion \( \iota : S \to M \) lifts to a marking \( \tilde{\iota} : S \to N \). Let \( \Phi \) denote the (isometric) deck transformation of \( N \) such that \( \tilde{\iota} \circ \phi \) is homotopic to \( \Phi \circ \tilde{\iota} \). Following the proof of [Bro03b, Theorem 1.1] there is a 1-Lipschitz map \( f : X = (S, g) \to N \) homotopic to \( \tilde{\iota} \) and a 1-Lipschitz sweepout \( f_t : X_t = (S, g_t) \to N \) from \( f_0 = f \) to \( f_1 = \Phi \circ f \circ \phi^{-1} \). (The hyperbolic structure \( X_1 \) on \( S \) agrees with that of \( X \) under \( \phi \), up to isotopy.) As in Theorem 2.1, this sweepout has the property that there is some curve \( \alpha \) in \( S \) such
that the geodesic representative of $\alpha$ in $N$ is in the image of $f$. Hence, the geodesic representative of $\phi(\alpha)$ lies in the image of $f_1$.

Let $H : S \times [0, 1] \to N$ be the homotopy given by $H(x, t) = f_t(x)$ and set $\Sigma_t$ to be the image of $f_t$. Finally, fix an embedding $h : S \to N$ homotopic to $\ell$ which lies to the left of the image of $H$. Note that there is some $n_0 \geq 1$ such that $\Phi^{n_0} h(S)$ lies to the right of the image of $H$.

For $n > 0$, define a function $s_n : [n, n + 1] \to [0, 1] by $s_n(x) = x - n$ and let $H^n : S \times [0, n] \to N$ denote the homotopy formed by gluing together

$$H, \quad \Phi \circ H \circ (\phi^1 \times s_1), \quad \ldots, \quad \Phi^{n-1} \circ H \circ (\phi^{-(n-1)} \times s_{n-1})$$

to form a sweepout from $f$ to $\Phi^n \circ f \circ \phi^{-n}$. (Note that $H^n$ is indeed continuous since the functions agree on their overlap.) Also, extend the definition of $\Sigma_t$ for $t \in [0, n]$ to be the image of $H^n(\cdot, t)$, so that in particular $\Sigma_n = \Phi^n(\Sigma_0)$. Note that the image of $H^n$ is contained in the compact region between $h(S)$ and $\Phi^{n+n_0}(h(S))$, which we name $C_n$.

To prove the first inequality, let $\rho : [0, l] \to M$ be the shortest loop in $M$ which realizes $\text{circ}_{c_\phi}(M)$. Note that $\rho$ cannot be $c_{\phi}$-short itself. Otherwise, since the image of $f$ under the covering $N \to M$ necessarily meets $\rho$, the argument in Lemma 3.3 would produce an essential loop in $S$ which is mapped into the Margulis tube about $\rho$. This would imply that $\rho$ represents an element of the kernel of $\pi_1(M_\phi) \to \mathbb{Z}$, a contradiction.

Denote by $\tilde{\rho}$ the preimage of $\rho$ in $N$ (joining the ends of $N$) and let $\tilde{\rho}_n = \tilde{\rho} \cap C_n$. Since $C_n$ is the union of $n + n_0$ fundamental domains of $\Phi$,

$$\ell_{c_N}^s(\tilde{\rho}_n) = (n + n_0) \cdot \ell_M^{c_s}(\rho).$$

By choice of $h(S)$ and $n_0$, each $\Sigma_t$ separates the boundary components of $C_n$ (which are $h(S)$ and $\Phi^{n+n_0} h(S)$) for $t \in [0, n]$. Hence, each such $\Sigma_t$ intersects $\tilde{\rho}_n$. Now pick any curve $\beta$ that is $L_\phi$-short on $X$ and observe that $\phi^n(\beta)$ is $L_\phi$-short on $X_n = \phi^n X$. Then, using Proposition 5.1 and Lemma 4.4 as in the proof of Theorem 4.1, we conclude that

$$d_{c(S)}(\beta, \phi^n(\beta)) \leq A_1(\|\chi(S)\|) \cdot \ell_{c_S}(\tilde{\rho}_n) \leq A_1(\|\chi(S)\|) \cdot (n + n_0) \cdot \ell_{c_N}^s(\rho).$$

Hence, diving both sides by $n$ and taking $n \to \infty$ shows that

$$\ell_S(\phi) \leq A_1(\|\chi(S)\|) \cdot \ell_M^{c_s}(\rho),$$

proving the first inequality.

For the second inequality, let $\xi_n$ be the shortest electric geodesic in $N$ joining the geodesic representatives of $\alpha$ and $\phi^n(\alpha)$, where $\alpha$ is as above. Apply Proposition 4.2 to these curves to obtain

$$\ell_{c_N}^s(\xi_n) \leq A_2(\|\chi(S)\|) \cdot d_{c(S)}(\alpha, \phi^n(\alpha)).$$

Alter $\xi_n$ to a new path $\omega_n$ as follows: for $0 < j < n$, choose some $x_j \in \xi_n \cap \Sigma_j$, and connect $x_j$ to $\Phi^j \tilde{\rho}(0) \in \Sigma_j$ by a shortest electric path $\gamma_j$ in $\Sigma_j$. For $j = 0$ and $j = n$, define $\gamma_j$ to be a shortest electric path in $\Sigma_j$ starting at the initial and terminal points of $\xi_n$ and ending at lifts $x_0$ and $x_n$ of $\rho(0)$ in $\Sigma_0$ and $\Sigma_n$, respectively. Then define $\omega_n$ to be the path obtained from $\xi_n$ by inserting $\gamma_j \ast \gamma_j^{-1}$ after $x_j$ for each $0 < j < n$,
and by inserting $\gamma_0^{-1}$ at the beginning and $\gamma_n$ at the end. Using the bounded diameter lemma, we have that

$$\ell^s_N(\omega_n) \leq \ell^s_N(\xi_n) + 2n \cdot \frac{4\pi|\chi(S)|}{\epsilon_S}$$

$$\leq \ell^s_N(\xi_n) + 2n \cdot A_2(|\chi(S)|).$$

Let $\omega_n[j - 1, j]$ denote the portion of $\omega_n$ between $\Phi^j\tilde{p}(0)$ and $\Phi^j\tilde{p}(0)$. Since $\omega_n[j - 1, j]$ descends to a loop in $M$ which is not in the kernel of $\pi_1(M) \to \mathbb{Z}$, we have

$$\ell^s_M(\rho) \leq \ell^s_N(\omega_n[j - 1, j]) \quad \forall j,$$

hence

$$n \cdot \ell^s_M(\rho) \leq \ell^s_N(\xi_n) + 2n \cdot A_2(|\chi(S)|)$$

$$\leq A_2(|\chi(S)|) \cdot d_{c(S)}(\alpha, \alpha^n) + 2n \cdot A_2(|\chi(S)|).$$

Dividing through by $n$ and taking a limit as $n \to \infty$ produces the second inequality.

\[\square\]

References


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