Semisimple Hopf actions on quantizations of Weyl algebras

joint w/ Juan Cuadra & Pavel Etingof

$k=\bar{k}$, ch(k)=0.

The goal: To understand examples of Hopf alg actions on algebras.

Given Hopf alg $H$, alg $A$, one could...

(what we'll actually do) Show that $H\otimes A$ factors through some algebra (possibly lots of quantization)

Classify all $H\otimes A$ (no quantization)

Hard because computations can be horrible

Depends on classification of Hopf algs.

Def: $H$ acts on $A$ if $A$ is an $H$-module algebra $\iff A$ is an algebra in $H$-mod.

If $H$ finite dim, there are two tractable classes of $H$

- Semisimple (easy)
- (as an $k$-alg) (all simple $H$-comodules are 1-dim)

...have a nice result when $A$ is a commutative domain.

\[ \text{Theorem:} \] Any semisimple Hopf action on a commutative domain factors through a group algebra action.

(Also see, if $H$ is a group algebra, then $H\otimes A$ is a Hopf algebra acting on $A$).

($k$-dim ideal $H_0$ acts faithfully on $\text{Ind}(H_0\otimes A)$)
We extend this result to semisimple Hopf-algebra actions on
quantum chains of commutative domains $B$

Take $B = \mathbb{A}_n(k)$ (Weyl algebra) = $k\langle x_1, y_1, \ldots, x_n, y_n \rangle / $ $\langle [x_i, y_j] = y_i, [y_j, x_i] = x_j, [x_i, x_j] = 0 \rangle$

Thm [Cuadra-Enrigue-W] Any semisimple Hopf-action on $\mathbb{A}_n(k)$

factors through a group algebra action.

[Che-Wang-W-Zheng] proved this result when

$\mathbb{A}_n(k)$ has the standard filtration

$H^2 \mathbb{A}_n(k)$ preserves the filtration of $\mathbb{A}_n(k)$

We don't assume this.

Proof follows in two steps: 1. Reduce $H^2 B = \mathbb{A}_n(k)$ modulo $P$( what B is a PID minus a com localise)

2. Study $H^2$ divisinably in positive char.

Skr 2

Proof: $H$-semisimple, a semisimple Hopf-algebra of dimension $D$

over an algebraically closed field $F$ of arbitrary characteristic.

$D$ = division algebra over $F$ of degree $N$ is $\mathbb{Z}(D)$

Then $\text{gcd}(D^N, N) = 1 \Rightarrow H^2 D$ factors through a group action.

BF degree argument with $\text{gcd}(D^N, N) = 1 \Rightarrow D = \mathbb{F}^{D^N}, \text{gcd} = \mathbb{Z}(D)$

$C_D(D^N) = \mathbb{Z}$

$\Rightarrow$ $\mathbb{Z}$ is $H$-stable ($H^1 \mathbb{Z}$) [can show $V \in \mathbb{E}$, $ht$, $\mathbb{A}(D^4) = 1$

Assume $H^2 D$ inner faithfully. Take $I =$ Hopf ideal of $H$ so that $I \cdot \mathbb{Z} = 0$

$D = \mathbb{F}^{D^N} \Rightarrow I \cdot D = 0 \Rightarrow I(0) \Rightarrow H^2 \mathbb{Z}$ (a field) inner faithfully

$\Rightarrow H$ is a group algebra.
Step 6

Reduction modulo p

Given $H \cong B$ inner-faithful

Goal: show for prime $p \gg 0$, fraction $H_p \cong B_p$ where $H_p$ action is outer $\overline{F_p}$

* Given $H^2 B = A_n(k)$ inner-faithful, $R^2 B = A_n(R)$

Here $H = R \otimes H^0$

* If homomorphism $\psi: R \to \overline{F_p}$ for $p \gg 0$ take $H_{p, \psi} = H \otimes R \otimes \overline{F_p} = : H_p$

* Get $H_p \cong B_p = A_n(\overline{F_p})$ for $p \gg 0$

Further, $H_p \cong B_p$ inner-faithful.

Proof of Theorem:

* Assume $H^2 B = A_n(k)$ inner-faithful

Reduce modulo p by step 6: $H_p \cong B_p$ inner-faithful

* Take $D_p = $ quotient division ring of $B_p$.

* [Skrigan-van Ostaegen] $H_p \cong D_p$ inner-faithfully

* Take $p > \dim H$, have $\deg (D_p) = p^n$

Proceed (step 2) $\Rightarrow H_p = H_{p, \psi}$ is co-commutative for any

homomorphism $\psi: R \to \overline{F_p}$

* Can Alg $\Rightarrow$ direct product of all such $\psi$ is an injection of $R$ into a direct product of fields

$\Rightarrow H_p$ is co-commutative

$\Rightarrow$ $H$ co-commutative if finite dim/field

$\Rightarrow H \cong R \otimes G$. //
What's next?

- Semi-simple Hopf alg actions on quantizations of con. domains $B$
  (to be posted later)

In the meanwhile, we have a question:

The technique of the proof of the main result worked because
for $B = A_n(t)$

- $B_p$ (the reduction of $B$ mod $p$) is a PID domain

$\star$ So we can localize to get divisibly $B_p$.

$\star$ So $\text{rad } B_p = p^n$ for $p > 0$

$\star$ So that we can employ Proposition $\star$ always

But in general:

**Question** Let $B$ be an $\mathbb{N}$-filtered algebra over $k$ with

- (B) a commutative domain finitely generated

$\star$ Field for $B$ for any large prime $p$??